World Journal of Economics and Business Research
ISSN: 2960-0081
DOI: https://doi.org/10.61784/wjebr3012

DYNAMIC OPTIMAL PORTFOLIO CHOICES FOR ROBUST PREFERENCES

JingShu Liu
Questrom School of Business, Boston University, Boston 02215, MA, US.
Corresponding Email: jingshul@bu.edu

Abstract: This paper solves the optimal dynamic portfolio choice problem for an ambiguity-averse investor. It introduces a new preference that allows for the separation of risk aversion and ambiguity aversion. The novel representation is based on generalized divergence measures that capture richer forms of model uncertainty than traditional the relative entropy measure. The novel preferences are shown to have a homothetic stochastic differential utility representation. Based on this representation, optimal portfolio policies are derived using numerical algorithms with forward-backward stochastic differential equations. The optimal portfolio policy is shown to contain new hedging motives induced by the investor's attitude toward model uncertainty. Ambiguity concerns introduce additional horizon effects, increase effective risk aversion, and overall reduce optimal investment in risky assets. These findings have important implications for the design of optimal portfolios in the presence of model uncertainty.
Keywords: Portfolio optimization; Ambiguity aversion; Robust-optimal control; Knightian uncertainty

1 INTRODUCTION

In paper, we study the optimal-portfolio problem of a long-term investor who faces model uncertainty and is ambiguity averse. We propose a robust-control criterion with a new utility formulation. We derive an equivalent stochastic differential utility (SDU) representation of the new robust preferences. For an investor with the constant relative risk aversion (CRRA) utility function, tractable optimal solutions are available from Schroder and Skiadas[1]. We present an alternative representation of the optimal solution in the setting of Ocone-Karatzas formula[2-3]. This representation decomposes the optimal portfolio into three parts: the mean-variance component, the dynamic hedging component for fluctuations in the stochastic investment opportunity set, and the hedging demand arising from robustness concerns. The mean-variance portfolio requires less investment in the stock market, compared with that in a setting without robustness concerns.

We provide the numerical implementations of the optimal solutions by solving a system of FBSDEs with the regression-based Monte Carlo approach[4]. We find that robustness concerns affect the pattern of the optimal portfolio. For the investor with CRRA utility and relative risk aversion greater (smaller) than one, ambiguity aversion increases (decreases) the inter-temporal hedging demand. The investor with logarithmic utility and robustness concerns is no longer myopic.

The question of dynamic optimal-portfolio allocation is of long-standing academic interest and practical importance. The mean-variance analysis proposed by Markowitz (1952) is the building block of the modern portfolio theory[5]. The seminal work of Samuelson (1969) and Merton (1971) suggests that the investor dynamically manages the optimal portfolio during the investment horizon[6-7]. Pliska (1986), Karatzas et al. (1987), and Cox and Huang (1989) propose a martingale approach that in a complete market setting, it allows to solve for the optimal consumption-investment plan as the solution to a static optimization problem by establishing the equivalence between the dynamic and static budget constraints[8-10]. Ocone and Karatzas (1991) derive explicit expressions for the hedging terms as conditional expectations of random variables related to the state dynamics[2]. Detemple et al. (2003) propose a simulation approach to calculate the dynamic hedging terms efficiently[3]. In these papers, the investor is modeled to live inside the world where the subjective and objective probability distributions coincide, that is, he/she knows the process that describes the state variables dynamics.

However, the Ellsberg (1961) experiments show that people's preferences are incompatible with the subjective expected utility in an environment with ambiguity, where the objective probability distribution does not agree with the subjective distributions. Ambiguity is sometimes referred to as "Knightian uncertainty", or in Hansen and Sargent (2001)'s terminology as "model uncertainty". In the financial market, one example of model uncertainty is the dynamics of the stock return process[11]. While the second moment of the return process can be estimated with reliable precision, the first moment (the expected return) is notoriously hard to estimate[12]. This difficulty to accurately infer the state dynamics induces ambiguity, and the possibility of model misspecification impacts how the investor designs the optimal portfolio.

Several models are stimulated to address the Ellsberg-based critiques: the multiple-priors model, the smooth ambiguity model and the multiplier utility model with the robust-control criterion introduced by Anderson et al. (2003) and Hansen and Sargent (2001), with later developments by Skiadas (2003), Maccheroni et al. (2006) and Skiadas (2013)[14-22]. These three sets of models have their respective merits. The first two models can rationalize the choice in the Ellsberg's experiments. The multiple-priors model can further exploit the qualitative differences from subjective expected utility, and the smooth ambiguity model has the advantage of separating "ambiguity" and "the attitude toward ambiguity". In the last model, optimal choices are observationally equivalent to those obtained with subjective expected utility. It also helps to solve the quantitatively puzzling price implications considering the subjective expected utility model. We cast our analysis in the robust-control setting, where the investor seeks a robust optimal strategy that performs best in the worst-case scenario of the model misspecification.

In this paper, we propose a robust-control problem, in which the investor trades off utility derived from consumption and the loss induced by ambiguity. Specifically, we introduce a new utility formulation that combines the consumption utility and ambiguity loss in a multiplicative way. The latter is quantified by a convex power function on the Radon-Nikodym derivative process, which is interpreted as a penalty for the discrepancy between the objective and subjective probability measures. Being concerned about the model uncertainty of the optimal plan, the investor decomposes the portfolio optimization procedure into two steps: 1) solving for the probability measure that generates the minimal expected utility, according to the degree of ambiguity aversion; 2) designing the optimal consumption-investment plan that maximizes the minimal utility, according to the degree of risk aversion.

There are three main contributions in this paper. First, we establish the equivalence between the robust-control problem with the new utility formulation and the SDU maximization problem. The equivalence between the Bellman equation for the robust-control criterion and that for the SDU specification has been noted by Hansen and Sargent (2001) and Maenhout (2004) in a different formulation of the utility function and state dynamics[11, 23]. With general state dynamics and consumption utility, we establish the same equivalence result. Skiadas (2003) establishes the equivalence when the discrepancy between the objective and subjective measures is quantified by the relative entropy[24]. The discrepancy measure we use here generalizes the relative entropy measure. In the case of CRRA consumption utility, our equivalence result can be viewed as a generalization of his result.

Second, we obtain closed-form optimal solutions for the robust-control problem for the investor with CRRA consumption utility. One set of such solutions is expressed by a system of FBSDEs as shown by Schroder and Skiadas (1999)[1]. We provide an alternative representation of the solution based on the refinement of the Ocone and Karatzas formula[1-2]. This representation helps us to decompose the optimal portfolio into three parts: the mean-variance component; the dynamic hedging component against fluctuations in the investment opportunity set; and the dynamic hedging demand from model uncertainty concerns.

The main impact of model uncertainty concerns is that it induces lower allocation to the risky asset in the mean-variance portfolio. Specifically, for the investor with CRRA utility, this impact is persistent along the investment horizon. For the investor with logarithmic utility, this impact vanishes with time, that is, the mean-variance portfolio gradually approaches the one without robustness concerns.

This impact helps us to understand the discrepancy between the degree of relative risk aversion implied by equilibrium asset prices and the value obtained from behavioral studies. Within the subjective expected utility models, the actual prices for macroeconomic risks are too high, as manifested by equity premium puzzles. This implies a high relative risk aversion. On the other hand, as concluded in Meyer and Meyer (2006) based on several studies on investors' behaviors, the relative risk aversion measure is typically small, in a range of one to four[25]. The investor's concerns for the model misspecification raise the prices for market risks and leads to a reinterpretation of the high prices as compensations for bearing model uncertainty. For the investor with CRRA utility, the impact of ambiguity averse is significant. For a moderate relative risk-aversion degree of 4, a penalty coefficient of -2 can adjust the relative risk aversion to 10. Conversely, this can be used to calibrate the fear for the model misspecification, as measured by the penalty coefficient of the ambiguity-averse investor. In a model with a constant investment opportunity set, Maenhout (2004) calibrates the penalty coefficient to be 14 with relative risk aversion as 7 to match the risk-free rate and equity premiums for 1981-1994, with an implied relative risk aversion of 21 without considering ambiguity aversion[23]. Such a difference suggests that the investors in the market indeed worry about model uncertainty and includes such concerns in the pricing of macroeconomic risks.

Finally, we provide numerical implementations for the dynamic optimal investment plan of the robust-control problem by solving systems of FBSDEs through the regression-based Monte Carlo method[4]. As calibrated by Campbell et al. (2003) in a model with a vector autoregressive return process, the dynamic hedging demand for an investor with recursive utility is substantial[26]. The numerical implementation helps us to gain insights into the inter-temporal hedging demand for fluctuations in the stochastic investment opportunity set and for ambiguity concerns. This also makes our work different from that of Maenhout (2004) who studies the optimal portfolio rule with a constant investment opportunity set, and the dynamic optimal portfolio implementation with subjective expected utility[2, 23].

We implement the optimal portfolio with the interest rate following an Ornstein-Uhlenbeck process, with the negative correlation with the stock market. The inter-temporal hedging demand boosts investment in the stock, which reflects the negative correlation between the two. The inter-temporal hedging demand decreases with time, representing the vanishing hedging need against fluctuations in the investment opportunity set and ambiguity averse. For the investor with constant relative risk aversion greater than or equal to one, the hedging demand increases (decreases) with ambiguity aversion.

In the setting where the interest rate follows an Ornstein-Uhlenbeck process, numerical results show that robustness concerns change the dynamic portfolio patterns for investors with different risk aversion. It is well known that in a setting without ambiguity, the investor with logarithmic utility does not have an inter-temporal hedging demand, even in the

stochastic investment environment. The model uncertainty concerns increase risk aversion, and thus introduces an inter-temporal hedging demand. With robustness concerns, the investor with logarithmic utility is no longer myopic.
In the setting with model uncertainty or ambiguity, comparative studies show that the optimal stock demand for an investor with constant relative risk aversion greater than one is larger for younger investors. This is consistent with the behavior that that younger investors invest more aggressively than older people.
Our work follows Skiadas (2003) which establishes the equivalence between the robust-control problem with the relative entropy formulation and the SDU maximization problem[24]. In the case of CRRA, our results can be viewed as an extension of his work. However, closed-form solutions are available in our setting for CRRA utility, whereas in his work, such solutions are only available for logarithmic utility. As we have discussed, optimal portfolios for these two utility functions have quite different dynamic patterns.
Maccheroni et al. (2006) propose and axiomatize an entropy-variational utility that unifies the multiple-prior utility and multiplier utility[27]. As shown in Skiadas (2013), the certainty equivalence based on this smooth divergence preference can be approximated by the expected-utility certainty equivalence, with the resulting recursive utility taking the form of an SDU[22]. Whereas these authors focus on the additive structure of the consumption utility and divergence loss, we look at the multiplicative structure and aim for closed-form optimal consumption-investment solutions.
Our work is related to Maenhout (2004) where he proposes a state-dependent penalty for the value function in the Bellman equation, in a setting of constant investment opportunities for an investor with CRRA utility[23]. Due to the specification of CRRA utility, the homothetic nature of the preference is maintained, and hence closed-form optimal solutions are available. Our work is different from his, in that the equivalence between our robust-control problem and the SDU maximization problem is established for the general form of consumption utility and dynamics of state variables. In addition, the optimal solution in our setting with the stochastic investment opportunity set includes inter-temporal hedging demands.
Chen et al. (2011) derives the dynamic portfolio choice solution in which the investor faces a model selection problem between an i.i.d return model and a vector autoregression model, with the recursive ambiguity utility[19, 28]. Maenhout (2006) extends Maenhout (2004) to a dynamic setting where the market price of risk is a mean-reversion process and derives the optimal portfolio through the dynamic programming approach[23,29]. Compared with their works, our work allows for general state-variable dynamics and inter-temporal consumption. Furthermore, the martingale-based approach we use in this paper allows us to obtain optimal solutions without having to use numerical schemes based on partial differential equations.
This paper is organized as follows. In Section 2, we introduce the robust control problem. In Section 3, we establish the equivalence between the robust-control problem and the SDU maximization problem. In Section 4, we provide an alternative optimal consumption-investment plan representation through the Ocone- Karazats formula. Section 5 provides numerical illustrations of the optimal portfolio. Section 6 concludes.

2 THE ROBUST-CONTROL PEROBLEM

2.1 The Background

We cast the analysis in a continuous-time model in which the underlying source of uncertainty is a d-dimensional Brownian motion B_t, $t \in [0,T]$. The probability space is (Ω, F, P), where P is the objective measure and the flow of information F_t, $t \in [0,T]$ is the filtration generated by the Brownian motion B_t. With limited knowledge about the objective probability measure, the investor's belief about the market can be modelled by a set of probability measures P^x equivalent to P.
Denote by $E(E^x)$ the expectation under $P(P^x)$, and E_t (E_t^x) the conditional expectation operator given F_t. Define the conditional density process d_t^x as $E_t\left[\frac{dP^x}{dP}\right]$, with the associated relative density defined as $d_{t,s}^x = \frac{d_s^x}{d_t^x}$.
By the martingale representation theorem, there exists an adapted process $x \in L_2$ such that:

$$d_t^x = \exp\left(\int_0^t x_s^{'} dB_s - \frac{1}{2}\int_0^t x_s^{'} x_s ds\right), \quad t \in [0,T]. \tag{1}$$

By the Girsanov theorem, the process B_t^x defined as $B_t^x = B_t - \int_0^t x_s ds$ is a Brownian motion under P^x. Consider a constant $\eta \in (-\infty, 1)$, define:

$$\tilde{d}_t = \exp\left(\frac{\eta}{\eta-1}\int_0^t x_s^{'} dB_s - \frac{\eta^2}{2(\eta-1)^2}\int_0^t x_s^{'} x_s ds\right), \tilde{B}_t = B_t - \frac{\eta}{\eta-1}\int_0^t x_s ds. \tag{2}$$

Set the random variable $\tilde{d} = \tilde{d}_T$. Define the probability measure $\tilde{P}$ as:

$$\tilde{P}(A) = \int_A \tilde{d}(\omega) dP(\omega), \ \forall A \in F. \tag{3}$$

Denote by $\tilde{E}(\tilde{E}_t)$ the (conditional) expectation operator under $\tilde{P}$.
We consider a complete market with a d-dimensional state variable Y_t and d risky securities. The state variable follows the vector-diffusion process $dY_t = \mu^Y(t, Y_t)dt + \sigma^Y(t, Y_t)dB_t$.

The investor allocates the wealth between the d risky securities and the money market account with the instantaneous risk-free rate $r_t = r(t, Y_t)$. The security prices $S_i, i = 1, \cdots, d$ follow the dynamics:

$$dS_{it} = S_{it}(\mu_i(t, Y_t)dt + \sigma_i(t, Y_t)dB_t),\ 1 \le i \le d, \quad (4)$$

where μ_i is the expected return process and σ_i is the vector of volatility coefficients of the i-th security. Denote by μ the d-dimensional vector of the expected returns, whose i-th entry is μ_i. Let σ denote the $d \times d$-dimensional volatility matrix whose rows are σ_i, $i = 1, \cdots, d$. Assume that σ is invertible. Also assume that μ and σ are progressively measurable and satisfy the standard integrability conditions. The market price of risk is defined as:

$$\theta_t = \theta(t, Y_t) \equiv \sigma(t, Y_t)^{-1}(\mu(t, Y_t) - r(t, Y_t)1),$$

where 1 is the d-dimensional unit vector. We assume that the market price of risk θ_t is continuously differentiable and satisfies the Novikov condition.

The state price density is defined as $\xi_t = \exp\left(-\int_0^t r_s ds - \int_0^t \theta_s' dB_s - \frac{1}{2}\int_0^t \theta_s'\theta_s ds\right)$ and the relative state price density is defined as $\xi_{t,s} = \frac{\xi_s}{\xi_t}$.

2.2 The Robust-Control Problem

In this section, we provide the definition of the robust-control problem and the utility formulation. The robust-control criterion is

$$\widehat{V}_t = \text{ess}\inf_x\{V_t^x\}, \qquad (2.1) \quad (5)$$

where the utility process V_t^x is defined as:

$$V_t^x = E_t\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right) u(c_s) \left(d_{t,s}^x\right)^{\frac{\eta}{\eta-1}} ds\right] = E_t^x\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right) u(c_s) \left(d_{t,s}^x\right)^{\frac{1}{\eta-1}} ds\right]. \quad (2.2) \quad (6)$$

The subjective discount factor β_t can be stochastic. The function $u(\cdot)$ is the real-valued Von Neumann-Morgenstern utility function. The penalty coefficient η represents the investor's averse attitude toward ambiguity. This attitude affects the utility function through the relative density process $d_{t,s}^x$. We assume:

$$0 < \eta < 1, \text{ if } u(\cdot) > 0,$$
$$-\infty \le \eta \le 0, \text{ if } u(\cdot) < 0. \quad (7)$$

The parameter ranges of η insure that it can model the investor's different degree of ambiguity aversion. In the limiting case of $\eta = -\infty$, the utility becomes $E_t^x[\int_t^T \exp(-\int_t^s \beta_v dv)u(c_s)ds]$. The penalty is so large that one cannot optimize the utility except for when the data-generating measure corresponds to P^x. In the case of $\eta = 0$, the utility is $E_t\left[\int_t^T \exp(-\int_t^s \beta_v dv)u(c_s)ds\right]$. The investor does not care about ambiguity. In between the two extremes is that the investor is averse to the divergence between P and P_x and tries to design a consumption-investment plan that performs best under the worst situation of the model misspecification.

The investor with ambiguity concerns is presented with a max-min problem. The first step in the optimization procedure is to find the probability measure under which the weighted expected utility is minimized. Next, we solve for the optimal consumption-investment plan that maximizes the minimized utility, subject to the dynamic budget constraint.

We interpret this two-step optimization problem as following. The investor trades off the gain from investment and the loss from ambiguity. The investment plan supports the contemporary consumption c_t, which induces utility. The conditional density d_t^x determines the loss induces by model uncertainty, i.e., the discrepancy between measures P and P_x. The coefficient η controls how severe the penalty the investor imposes for not knowing the true model. If the penalty is too high, the investor can have a very conservative plan. If the penalty is too low, the investor is exposed to model uncertainty and the optimal plan may perform worse than what the control theory has promised.

Borrowing the idea of a generalized loss function form Berger (1985), we specify the (integrated) loss function $L(c_t, d_t^x)$ as[30]:

$$L(c_t, d_t^x) = u(c_t)(d_t^x)^{\frac{1}{\eta-1}}. \quad (8)$$

The multiplicative formulation captures the fact that the impact of ambiguity concern is dependent on the contemporary consumption utility. For an investor with low-consumption utility, even though he/she is aware of and imposes a high penalty for ambiguity, such concerns do not affect the total utility as much as compared with an otherwise identical investor with high-consumption utility.

As we shall see, this utility formulation with the CRRA utility $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$ helps to model ambiguity aversion as a penalty component on the utility process. In contrast to the relative entropy formulation, this proportion is state dependent. It also allows us to convert the robust-control problem to a class of homothetic SDU where closed-form solutions for the optimal consumption-investment plan are available[1]. Besides this desirable tractability property, insights gained from the optimal-portfolio solution reveal that ambiguity aversion can help reconciling the high-risk aversion implied by asset prices and the moderate degree obtained from behavioral studies.

This utility formulation also allows for dynamic optimal-portfolios solutions with inter-temporal hedging components against fluctuations in the investment opportunity set and ambiguity aversion. Tractable solutions are also available for the relative-entropy formulation, but only when the investor has logarithmic utility, which is clearly a limitation.

3 MAIN RESULTS

Our main result establishes the connection between the robust-control problem (2.1) and a form of the SDU problem.
Theorem 3.1. There exists a unique progressively measurable pair (V, σ^b), such that:

$$dV_t = -\left(u_t - \beta V_t - \frac{\eta}{2V_t}(\sigma_t^b)'\sigma_t^b\right)dt + (\sigma_t^b)'dB_t, \quad (3.1) \tag{9}$$

with the boundary condition $V_T = 0$.
For $t \in [0, T)$ and any adapted process x, the utility process V_t^x can be expressed as:

$$V_t^x - V_t = \tilde{E}_t\left[\int_t^T \exp\left(\frac{\eta}{2(\eta-1)^2}\int_t^s x_v' x_v dv\right) \times \left(\frac{1}{2}\left(x_s\sqrt{\frac{\eta}{(\eta-1)^2}V_s} + \sigma_s^b\sqrt{\frac{\eta}{V_s}}\right)'\left(x_s\sqrt{\frac{\eta}{(\eta-1)^2}V_s} + \sigma_s^b\sqrt{\frac{\eta}{V_s}}\right)\right)ds\right]. \tag{10}$$

The utility process V_t^x is minimized at:

$$x^* = -\sigma_t^b\left|\frac{\eta-1}{V_t}\right|, \tag{11}$$

and

$$V^{x^*} = V. \tag{12}$$

The existence result of (V_t, σ_t) in Equation (3.1) can be found in Schroder and Skiadas (1999)[1].
Proof. See Appendix A.
With the optimizing value of x^*, we approach the robust-control problem (2.1) by solving (V_t, σ_t) described by the backward stochastic differential equation (BSDE) in Equation (3.1). We can see from this equation that ambiguity concerns introduce a state-dependent penalizing component on the utility process.
The next result shows that the BSDE in Equation (3.1) can be expressed in the form of the SDU. We define an ordinally equivalent utility process of V_t, which allows us to express the preference in terms of a homothetic SDU:

$$v_t = \begin{cases} V_t^{1-\eta} \text{ for } V_t > 0, \\ -(-V_t)^{1-\eta} \text{ for } V_t < 0, \end{cases} \tag{13}$$

with the boundary condition $v_T = 0$.
Proposition 3.2. The transformed utility process v_t can be expressed as:

$$v_t = E_t\left[\int_t^T (1-\eta)\left(|v_s|^{\frac{\eta}{\eta-1}}u_s - \beta v_s\right)ds\right]. \quad (3.2) \tag{14}$$

Proof. See Appendix A.
Given the special case of the CRRA utility function, Equation (3.2) corresponds to the homothetic SDU specification in Schroder and Skiadas (1999) with $\alpha = -\eta$, and $\gamma \neq 1$. In that context, the coefficient α is interpreted as a measure of the preference for the timing of uncertainty resolution.
Skiadas (2003) obtains similar results in the relative-entropy formulation, which is a special case of ours[24]:

$$V_t^x = E_t^x\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right)\log(c_s)\,ds\right] + \frac{1}{2\eta}E_t^x\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right)x_s^2 ds\right]. \quad (3.3) \tag{15}$$

The second term is the relative entropy distance considered in Hansen and Sargent (2001) and Skiadas (2003), with $\frac{1}{\eta}$ being the penalizing coefficient[11, 24]. Skiadas (2003) shows that the robust-control problem can be solved by a BSDE for (V_t, σ_t):

$$dV_t = -\left(\log(c_t) - \beta V_t - \frac{\eta}{2}\sigma_t^2\right)dt + \sigma_t dB_t, \tag{16}$$
$$V_T = 0.$$

We can express the solution to this BSDE as a form of the homothetic SDU. To do this, we define an ordinally equivalent utility process v_t of V_t as:

$$v_t = -\frac{1}{\eta}(\exp(-\eta V_t) - 1), \tag{17}$$

with the boundary condition $v_T = 0$.
Proposition 3.3. The transformed utility process v_t can be expressed as:

$$v_t = E_t\left[\int_t^T (1-\eta v_s)\left(\log(c_s) + \frac{\beta}{\eta}\log(1-\eta v_s)\right)ds\right]. \quad (3.4) \tag{18}$$

To conclude, the utility form (3.2) corresponds to the homothetic SDU specification in Schroder and Skiadas (1999), with $\alpha = -\eta$ and $\gamma = 1$. In this case, closed-form solutions for the optimal consumption-investment plan are available[1]. Comparing our formulation with that of Skiadas (2003), closed-form solutions for the optimal plan are not available in their formulation for CRRA utility[24]. Our robust-control problem has closed-form solutions for CRRA utility and for logarithmic utility as a special case.

4 THE OPTIMAL CONSUMPTION-INVESTMENT PLAN

In this section, we provide an alternative representation of the optimal solution of the robust problem (2.1) for an investor with CRRA utility based on the generalized Clark-Ocone formula of the hedging terms using Malliavin calculus[2-3]. We also provide the optimal-portfolio solution for the problem (3.3) as a special case.

Detemple et al. (2003) express the hedging terms by conditional expectations with respect to the Malliavin derivatives and propose a simulation-based approach for the subjective expected utility optimization problem with a stochastic investment opportunity set[2]. Here we adopt their methodology to solve for the optimal portfolio in the SDU maximization problem setup.

From this new representation, we can separate the forward-hedging component against fluctuations in the market price of risk and interest rate, and the backward-hedging component for ambiguity concerns.

4.1 The Optimal Solution: the CRRA Utility Case

For the investor with ambiguity concerns, after finding the probability measure under which the expected utility has the minimal value, he/she seeks to maximize the expected utility by selecting the optimal consumption-investment plan subject to a dynamic budget constraint:

$$\max_{\{\pi_t, c_t\}} E_0\left[\int_0^T \left(1-\frac{1}{\eta}\right)\left(|v_s|^{\frac{1}{1-\eta}}u_s - \beta v_s\right)ds\right], \quad (4.1) \tag{19}$$

s.t

$$dw_t = (r_t w_t - c_t)dt + w_t\pi_t^{'}\left((\mu_t - \mathbf{1}r_t)dt + \sigma_t dB_t\right), \;\; w_0 = w, \tag{20}$$
$$w_t \geq 0, \forall\, t \in [0,T].$$

Here w_t is the investor's wealth process at time t and w is the initial wealth. The term π_t is the proportion invested in the risky assets at time t. The nonnegativity constraint is the typical no-bankruptcy condition. The zero lower boundary can be replaced by a finite negative value. The utility function $u(\cdot)$ satisfies the assumption of strictly increasing and concave, with limits $\lim_{x\to\infty} u^{'}(x) = 0$ and $\lim_{x\to\infty} u^{''}(x) < \infty$. For the problem (4.1) with the CRRA utility function $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, Schroder and Skiadas (1999) derive explicit solutions for the optimal consumption, the utility process and the optimal portfolio. Before going to our results, we present the main results of Schroder and Skiadas (1999) to introduce definitions and notations[1].

4.2 Schroder and Skiadas (1999): the CRRA Utility Case[1]

(1)The optimal consumption:

Denote $\alpha = -\eta$, the optimal consumption is

$$c_t = (1+\alpha)^{\frac{1}{\gamma}}|v_t|^{\frac{\alpha k}{1+\alpha}}\exp\left(-\frac{X_t}{\gamma}\right) = (1+\alpha)^{\frac{1}{\gamma}}\exp\left(-kX_t\right)|J_t|^{\alpha k}, \tag{21}$$

with

$$k = \frac{1}{1-(1-\gamma)(1+\alpha)}. \tag{22}$$

(2)The optimal portfolio is:

$$\sigma_t\pi_t = k\theta_t + (1+\alpha k)\frac{Z_t}{J_t}. \tag{23}$$

(3)The pair (J_t, Z_t) and X_t:

The backward component (J_t, Z_t) and the forward component X_t together solve the FBSDE system:

$$dJ_t = -\left(\frac{1}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}} + \frac{1-\gamma}{\gamma}\left(r_t - \frac{\beta}{1-\gamma} + \frac{k\theta_t^{'}\theta_t}{2}\right)J_t + \frac{\alpha k}{2}\frac{Z_t^{'}Z_t}{J_t}\right)dt + Z_t^{'}(dB_t + (1-k)\theta_t dt),$$
$$J_T = 0,$$
$$dX_t = -\left(\frac{\alpha}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}}J_t^{-1} - (1+\alpha)\beta + r_t + \frac{\theta_t^{'}\theta_t}{2}\right)dt - \theta_t^{'}dB_t, \tag{24}$$
$$X_0 = \log(\lambda).$$

The value of λ is obtained by imposing the static budget constraint:

$$E_0\left[\int_0^T \xi_s c_s ds\right] = \omega. \tag{25}$$

The forward state-variable dynamics (r_t, θ_t) and the backward process (J_t, Z_t) solve an FBSDE system. The process X_t is the logarithm of the state price density, adjusted for ambiguity concern by including the backward component J_t.

The next theorem compiles the forward dynamics (the state variables and their Malliavin derivatives) with the backward dynamics (the process J_t and its Malliavin derivatives) into an FBSDE system. Numerical methods to solve this system are available. We provide the numerical illustration in the next section. The theoretic foundation of Malliavin calculus can be found in Nualart (1995), with the application to finance found in Karatzas et al. (1987), Karoui et al. (1997), and Detemple et al. (2003)[2-3, 31-32].

Theorem 4.1. The dynamics of J_t in the CRRA utility case is given by Schroder and Skiadas (1999)[1]. The processes J_t and D_sJ_t, $0 \le s \le t \le T$ can be solved via the (decoupled) FBSDE system[33-34]:

Forward dynamics:

$$dx_t = b(x_t)dt + \sigma(x_t)dB_t, \tag{26}$$

where

$$x_t = \begin{pmatrix} Y_t \\ D_sY_t \end{pmatrix}, b_t = \begin{pmatrix} \mu(t, Y_t) \\ \partial_2\mu(t, Y_t)D_sY_t \end{pmatrix}, \sigma(x_t) = \begin{pmatrix} \sigma(t, Y_t) \\ \partial_2\sigma(t, Y_s)D_sY_t \end{pmatrix}, \tag{27}$$

and the initial conditions are:

$$x_0 = Y_0, D_sY_s = \sigma(s, Y_s). \tag{28}$$

Backward dynamics:

$$-dy_t = f(s, t, x_t, y_t, z_t)dt - z_tdB_t, \tag{29}$$

with

$$y_t = \begin{pmatrix} J_t \\ D_sJ_t \end{pmatrix}, z_t = \begin{pmatrix} Z_t \\ D_sZ_t \end{pmatrix}, \tag{30}$$

and

$$f(s, t, x_t, y_t, z_t) = \begin{pmatrix} \frac{1}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}} + \frac{1-\gamma}{\gamma}\left(r_t - \frac{\beta}{1-\gamma} + \frac{k\theta_t'\theta_t}{2}\right)J_t + \frac{\alpha k}{2}\frac{Z_t'Z_t}{J_t} - Z_t'(1-k)\theta_t \\ \frac{1-\gamma}{\gamma}\left(\left(D_sr_t + k\theta_t'D_s\theta_t\right)J_t + \left(r_t - \frac{\beta}{1-\gamma} + \frac{k\theta_t'\theta_t}{2}\right)D_sJ_t\right) \\ + \frac{\alpha k}{2}\left(\frac{2\,Z_t'}{J_t}D_sZ_t - \frac{Z_t'Z_t}{J_t^2}D_sJ_t\right) - (1-k)(\theta_t'D_sZ_t + Z_t'D_s\theta_s) \end{pmatrix}, \tag{31}$$

with the following definitions of D_sr_t and $D_s\theta_t$:

$$D_sr_t = \partial_2r(t, Y_t)D_sY_t, D_s\theta_t = \partial_2\theta(t, Y_s)D_sY_t. \tag{32}$$

The boundary conditions are:

$$J_T = 0, D_sJ_T = 0 \times 1'. \tag{33}$$

Proof. See Appendix A.

Proposition 4.2. With the solution of (J_t, D_sJ_t), the pair (X_t, D_sX_t) can be solved by the system:

$$\begin{gathered} dX_t =- \left[\frac{\alpha}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}}J_t^{-1} - (1+\alpha)\beta + r_t + \frac{\theta_t'\theta_t}{2}\right]dt - \theta_t'dB_t, \\ X_0 = \log(\lambda), \\ dD_sX_t =- \left[D_sr_t - \frac{\alpha}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}}J_t^{-2}D_sJ_t\right]dt - (dB_t + \theta_tdt)'D_s\theta_t. \\ D_sX_s =- \theta_s'. \end{gathered} \tag{34}$$

Proof. See Appendix A.

Theorem 4.3. We provide an alternative representation of the optimal-portfolio solution by the Clark-Ocone formula. Denote by $H_{t,s}$ the inter-temporal hedging demand against fluctuations in the investment opportunity set with the expression:

$$H_{t,u} = \int_t^u D_tr_vdv + \int_t^u (dB_v + \theta_vdv)'D_t\theta_v. \tag{35}$$

Denote $X_{t,u}$ as $X_u - X_t$. The optimal portfolio is

$$w_t\pi_t^{'}\sigma_t = E_t\left(\int_t^T \xi_{t,u}c_u\left(-kD_tX_{t,u} + \frac{ak}{J_u}D_tJ_u\right)du\right) + k\theta_t w_t - E_t\left(\int_t^T \xi_{t,u}c_uH_{t,u}du\right) = kw_t\theta_t^{'} + (k-1)E_t\left(\int_t^T \xi_{t,u}c_uH_{t,u}du\right) + E_t\left(\int_t^T \xi_{t,u}c_u\left(\frac{\alpha k}{J_u}D_tJ_u - \frac{k\,\alpha(1+\alpha)^{\frac{1-\gamma}{\gamma}}}{1-\gamma}\int_t^u \frac{D_tJ_v}{J_v^2}dv\right)du\right). \tag{36}$$

Proof. See Appendix A.

The optimal portfolio is decomposed into three parts. The first component is the mean-variance portfolio. The second component is the hedging demand against fluctuations in the investment opportunity set. The third hedging comes from robustness concerns[35].

Theorem 4.4. The backward component J_t and its Malliavin derivative D_sJ_t, $0 < s < t$ can be solved via the (decoupled) FBSDE system:

Forward dynamics:

$$dx_t = b(x_t)dt + \sigma(x_t)dB_t. \tag{37}$$

The forward system is the same as that in Theorem (4.1).

Backward dynamics:

$$-dy_t = f(s,t,x_t,y_t,z_t)dt - z_tdB_t, \tag{38}$$

with

$$y_t = \begin{pmatrix} J_t \\ D_sJ_t \end{pmatrix}, z_t = \begin{pmatrix} Z_t \\ D_sZ_t \end{pmatrix}, \tag{39}$$

and

$$f(s,t,x_t,y_t,z_t) = \begin{pmatrix} (1-k_t)\left(\beta - r_t - \frac{k_t\theta_t^{'}\theta_t}{2} - Z_t^{'}\theta_t\right) + k_t(\alpha-\beta)J_t + \frac{1}{2}Z_t^{'}Z_t \\ (1-k_t)\left(-D_sr_t - k_t\theta_t^{'}D_s\theta_t - Z_t^{'}D_s\theta_t - \theta_t^{'}D_sZ_t\right) + k_t(\alpha-\beta)D_sJ_t + Z_t^{'}D_sZ_t \end{pmatrix}. \tag{40}$$

The boundary conditions are:

$$J_T = 0, D_sJ_T = 0 \times 1^{'}. \tag{41}$$

Proof. See Appendix A.

Proposition 4.5. With the solution of (J_t, D_sJ_t), the pair (X_t, D_sX_t) can be solved by:

$$X_t = -\int_0^t e^{-\int_s^t((\beta-\alpha)k_v-\beta)dv}\left((\alpha-\beta)J_s - \beta + r_s + \frac{\theta_s^{'}\theta_s}{2}\right)ds - \int_0^t e^{-\int_s^t(\beta-\alpha)k_v-\beta)dv}\theta_s^{'}dB_s + e^{-\int_0^t((\beta-\alpha)k_v-\beta)dv}\log(\lambda), \tag{42}$$

$$D_sX_t = -\int_s^t e^{-\int_v^t((\beta-\alpha)k_l-\beta)dl}\left((\alpha-\beta)D_sJ_v + D_sr_v)dv + (\theta_vdv + dB_v)^{'}D_s\theta_v\right) - e^{-\int_s^t((\beta-\alpha)k_l-\beta)dl}\theta_s^{'}.$$

Proof. See Appendix A.

With Proposition 4.5, we can decompose the hedging demand for X_t into the hedging demand against fluctuations in the investment opportunity set and that related to the backward term J_t. This enables us to express the optimal portfolio as three components: the mean-variance portfolio, the forward-hedging term related to D_sX_t and the backward-hedging term related to D_sJ_t.

Theorem 4.6. The optimal portfolio can be expressed as:

$$\begin{aligned} w_t\pi_t^{'}\sigma_t &= E_t\left[\int_t^T (D_tJ_u - k_uD_tX_{t,u})c_u\xi_{t,u}du\right] - E_t\left[\int_t^T c_u\xi_{t,u}H_{t,u}du\right] + \theta_t^{'}E_t\left[\int_t^T c_u\xi_{t,u}k_udu\right] \\ &=- E_t\left[\int_t^T c_u\xi_{t,u}H_{t,u}du\right] + \theta_t^{'}E_t\left[\int_t^T c_u\xi_{t,u}k_udu\right] \\ &- E_t\left[\int_t^T\left(\int_t^u e^{\int_v^t((\beta-\alpha)k_l-\beta)dl}\left(D_tr_vdv + (dB_v + \theta_vdv)^{'}D_t\theta_v\right)dv\right)k_uc_u\xi_{t,u}du\right] \\ &+ E_t\left[\int_t^T\left(D_tJ_u + k_u\left(\int_t^u e^{\int_v^t((\beta-\alpha)k_l-\beta)dl}\left((\alpha-\beta)D_tJ_v\right)dv\right)\right)c_u\xi_{t,u}du\right]. \end{aligned} \tag{43}$$

Proof. See Appendix A.

5 NUMERICAL RESULTS

In the numerical experiments, we assume that the short rate follows an Ornstein-Uhlenbeck process as in the Vasicek model:

$$dr_t = (\alpha_r - \beta_r r_t)dt + \sigma_r dw_t. \qquad (5.1)$$

Parameters used for the numerical illustration are summarized in Table 1. Before presenting the numerical results for optimal solutions, we first illustrate the performance of the regression-based Monte Carlo method in Gobet et al. (2005) applied to solve the FBSDE systems[36].

Table 1 Parameters Specifications

T	h	M	w	r_0	α_r	β_r	σ_r	θ	σ_s	β	γ
25	1/12	10000	20	0.07	0.2	4	-0.12	0.09	0.33	0.01	4

*Note: This table reports parameter values for numerical implementations. T: the investment horizon; h: the discretization step; M: the number of trajectories for the Monte Carlo simulation; w: the initial wealth; $r_0, \alpha_r, \beta_r, \sigma_r$: parameters in the Vasicek dynamics of the short rate process; θ: the market price of risk; β: the subjective discount factor; σ_s: the stock volatility; γ: the coefficient of relative risk aversion.

5.1 The Performance of the Regression-Based Method

In this section, we use the regression-based method proposed by Gobet et al. (2005) to solve the FBSDE systems of (X_t, D_sX_t) and $(Y_t, D_sY_t, J_t, D_sJ_t)$ for CRRA and logarithmic utility[4]. We introduce the following notations:

$$H^1_{s,t,T} = -\int_t^T (1-k_v)\left((1-k_v)^2\theta_v' D_s\theta_v\right)dv - \int_t^T (1-k_v)dB_v' D_s\theta_v,$$
$$H^2_{s,t,T} = -\int_t^T (1-k_v)\left(D_s r_v + k_v\theta_v' D_s\theta_v\right)dv, \qquad (44)$$
$$H^3_{s,t,T} = \frac{1-\gamma}{\gamma}\int_t^T \left(D_s r_u + k\theta_u' D_s\theta_u\right)du.$$

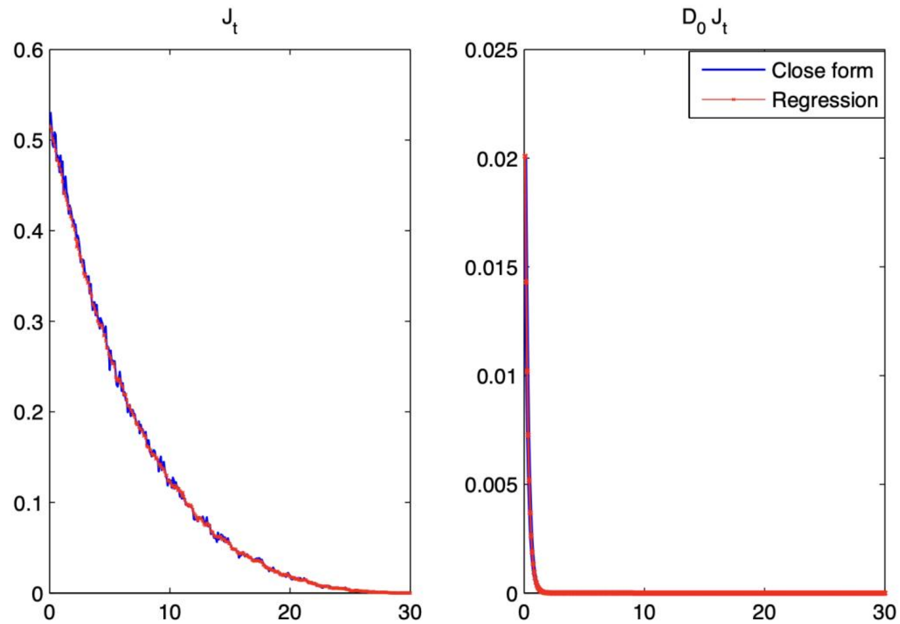
In the case of logarithmic utility, with parameters constraints $\alpha = 0$ or $\alpha = \beta$, the processes J_t and $D_sJ_t (s < t)$ can be solved explicitly:

$$\exp(J_t) = E_t\left[\tilde{\xi}_{t,T}\exp\left(\int_t^T (1-k_v)\left(\beta - r_v - \frac{k_v}{2}\theta_v'\theta_v\right)dv\right)\right],$$
$$\tilde{\xi}_{t,T} = \exp\left(-\int_t^T (1-k_v)\theta_v' dB_v - \int_t^T \frac{(1-k_v)^2\theta_v'\theta_v}{2}dv\right), \qquad (45)$$
$$D_sJ_t = \frac{E_t\left[\tilde{\xi}_{t,T}\exp\left(\int_t^T (1-k_v)\left(\beta - r_v - \frac{k_v}{2}\theta_v'\theta_v\right)dv\right)(H^1_{s,t,T} + H^2_{s,t,T})\right]}{J_t}.$$

The first two equations are from Schroder and Skiadas (1999)[1]. The last equation is obtained by applying the chain rule of Malliavin calculus. In the case of $\alpha = 0$, the values of J_t and D_sJ_t are both zero.

We apply the Monte Carlo simulation method to compute the conditional expectations, which are used as a benchmark to evaluate the performance of the regression-based method. As shown in Figure 1, the regression-based method generates numerical solutions that are very close to those calculated from Monte Carlo simulations.

Figure 1 In the Case of $\alpha = \beta$, the Processes J_t and D_0J_t for Logarithmic Utility have Closed-Form Solutions as Conditional Expectations



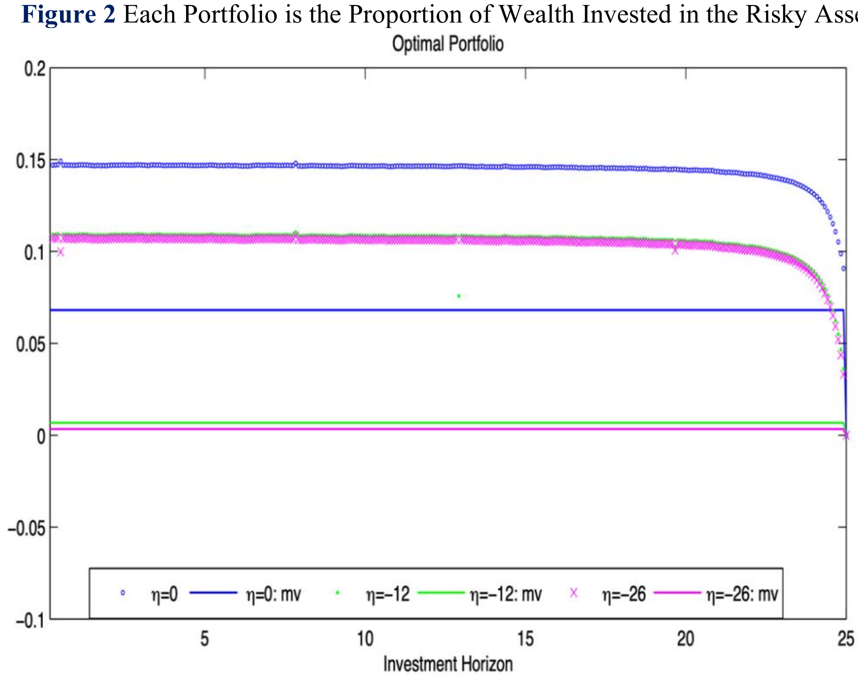
*Note: This figure reports the numerical solutions for J_t and D_0J_t obtained using the regression-based approach and the Monte Carlo simulation approach.

5.2 The Optimal Portfolio

In this section, we study the impacts of ambiguity aversion on the dynamic optimal portfolio in a setting with stochastic investment opportunities. Specifically, we assume a stochastic interest rate process with the dynamics in the Equation (5.1) and keep the market price of risk θ as constant. The latter can also be specified as stochastic.

We see from the Figure 2 that the optimal-portfolio pattern shares some common features with those obtained in an environment without ambiguity. The hedging component changes sign as relative risk aversion is in excess or falls short of one. This illustrates the knife-edge behavior of logarithmic utility. For the investor with relative risk aversion greater than one, the hedging demand for interest rate fluctuations boosts demand fors risky assets, due to the negative correlation between the interest rate and the stock price.

Figure 2 Each Portfolio is the Proportion of Wealth Invested in the Risky Asset



*Note: This figure plots the portfolios for the investor with CRRA utility of $\gamma = 4$ and different ambiguity penalty coefficients.

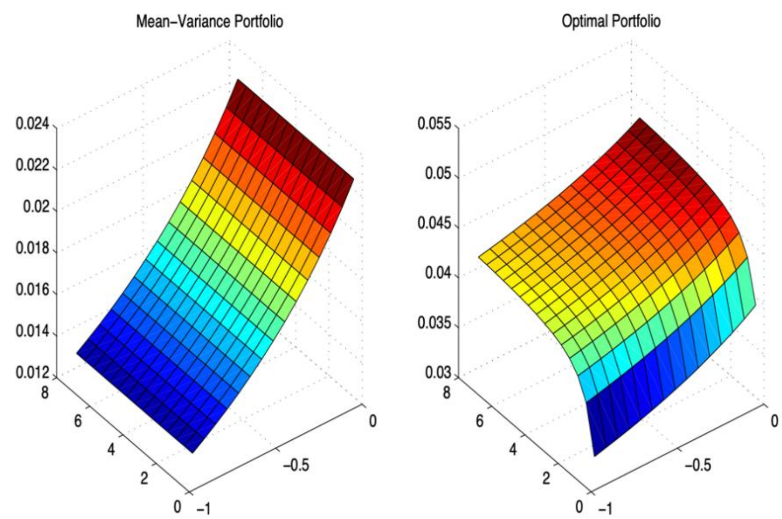
More interestingly, we see significant impacts on the dynamic optimal portfolio from ambiguity aversion. Figure 2 illustrate that ambiguity aversion equivalently increases risk aversion, in a sense that it lowers the constant mean-variance portfolio. The adjusted relative risk aversion is $\gamma + (\gamma - 1) \times (-\eta)$. To reconcile the high relative risk aversion implied by asset prices (for instance, 10) with the moderate degree implied by behavioral studies (for instance, 4), we need an ambiguity penalty coefficient of -2.

In addition to the mean-variance portfolio, ambiguity concerns decrease the total proportion of wealth invested in the stock market. The higher the penalty the investor imposes, the less aggressively he/she invests in the stock market.

In a setting with no ambiguity, the investor with logarithmic utility only invests in the mean-variance portfolio even with a stochastic investment opportunity set. The pattern of the optimal portfolio changes when this investor is ambiguity averse. First, the mean-variance portfolio is time-dependent, even with a constant market price of risk. It is lower at the initiation of investment, as ambiguity aversion increases risk aversion. It grows and approaches the mean-variance portfolio obtained in a setting with no ambiguity at the end of the investment horizon. Second, the investor is no longer myopic and requests an inter-temporal hedging demand. The hedging demand is positive, as it contains the component to hedge for fluctuations in the interest rate, which is negatively correlated with the stock market.

Figure 3 displays the behavior of the optimal portfolio and its mean-variance component relative to the ambiguity aversion and investment horizon, for an investor with CRRA utility and relative risk aversion of 4. The investment horizon is from 0.5 to 7.5 years and the ambiguity penalty is from -1 to 0. The higher the absolute value of the penalty, the more ambiguity averse the investor is. As expected, ambiguity inversion shifts the mean-variance portfolio toward a lower level. The mean-variance portfolio is constant over the investment horizon. The total portfolio, as a fraction of wealth invested in the stock market, is a decreasing (increasing) function of ambiguity aversion (the investment horizon).

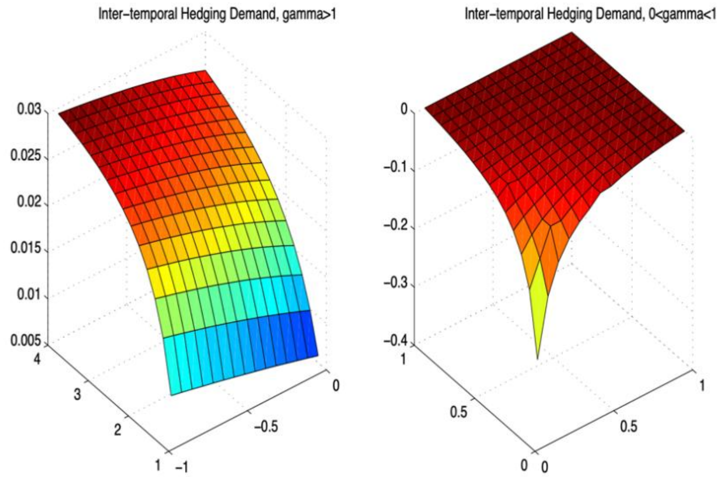
Figure 3 The Optimal Portfolio is the Proportion of Wealth Invested in the Risky Asset



*Note: This figure plots the optimal portfolios for the investor with CRRA utility of $\gamma = 4$, with different ambiguity penalty and investment horizons.

Figure 4 shows the effects of ambiguity aversion and risk aversion on the inter-temporal hedging demand. For an investor with relative risk aversion greater than one, the hedging demand increases with risk aversion and/or ambiguity aversion. On the contrary, when the investor has relative risk aversion smaller than one, this hedging demand is decreasing with risk aversion and/or ambiguity aversion. When the ambiguity aversion and/or risk aversion approach one, the hedging term vanishes. These facts suggest that from the perspective of the hedging demand, to include ambiguity aversion is also observationally equivalent to increasing risk aversion.

Figure 4 The Optimal Portfolio is the Proportion of Wealth Invested in the Risky Asset

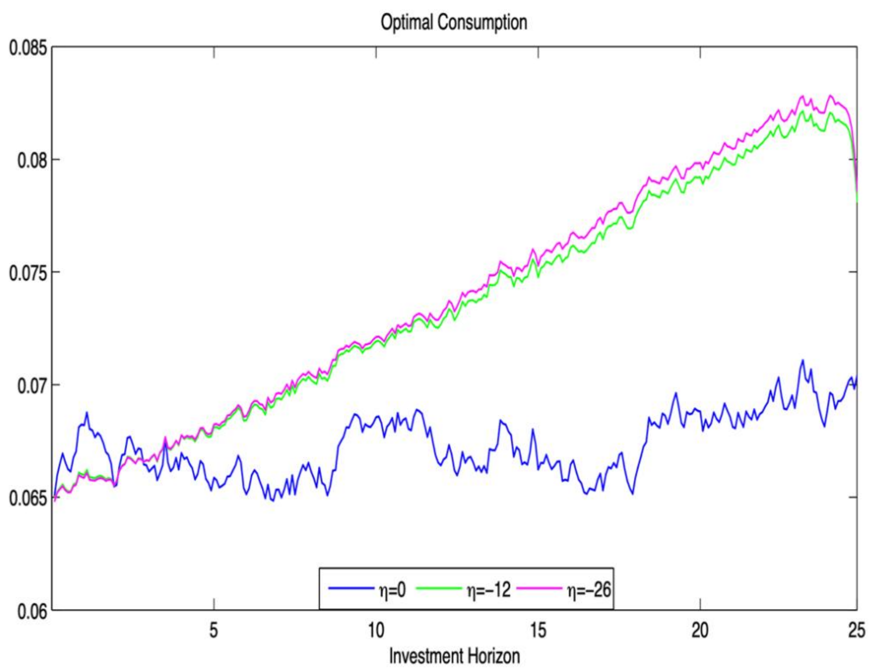


*Note: This figure plots the inter-temporal hedging demands for the investor with CRRA utility of the investment length $T = 15$, with different ambiguity penalty coefficients and risk version coefficients.

5.3 The Optimal Consumption

As illustrated in Figure 5, robustness concerns have a significant effect on the investor's consumption process. The consumption level grows with age as wealth accumulates with time. When the investor ignores ambiguity (i.e., $\eta = \infty$), the optimal consumption level increases as risk aversion increases. The reason is that the more risk averse the investor is, the smoother the consumption plan he/she prefers. As a result, he/she invests less and consumes more. Ambiguity aversion increases risk aversion, and thus induces a higher and more smoothed consumption path.

Figure 5 Optimal Consumption Expenditures are Normalized by the Initial Wealth



*Note: This figure plots the optimal consumption plans for the investor with CRRA utility of $\gamma = 4$ and different ambiguity penalty coefficients.

6 CONCLUSIONS

In this paper, we propose a robust-control problem with a new utility formulation, in which the investor trades off the multiplicative structure of utility derived from consumption and the loss from ambiguity.

We establish the equivalence between this robust-control problem and the SDU maximization problem. Insights obtained from this equivalence result show that the investor with robustness concerns prefers early resolution of uncertainty. We obtain closed-form optimal solutions when the investor has CRRA utility. We provide an alternative representation of the optimal solution based on the Ocone and Karatzas formula[2-3]. This representation decomposes the optimal portfolio into three parts: the mean-variance component, the dynamics hedging component against fluctuations in the investment opportunity set, and the dynamic hedging component for ambiguity concerns.

Numerical implementations for the dynamic optimal solution are provided for the robust-control problem (equivalently, the SDU maximization problem) through the regression-based method for solving the FBSDE systems. This helps us to gain insights into the inter-temporal hedging demand. For the investor with the relative risk aversion greater (smaller) than one, the hedging demand increases (decreases) with ambiguity aversion. With robustness concerns, the investor with logarithmic utility is no longer myopic and has an inter-temporal hedging demand. For the ambiguity-averse investor with constant relative risk aversion greater than one, the younger the investor is, the more aggressively he/she invests in the stock market.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

REFERENCES

[1] M Schroder, C. Skiadas. Optimal consumption and portfolio selection with stochastic differential utility. Journal of Economic Theory, 1999, 89(1): 68-126.

[2] DL Ocone, I Karatzas. A generalized Clark representation formula, with application to optimal portfolios. Stochastics and Stochastic Reports, 1991, 34(3-4): 187-220.

[3] J Detemple, R Garcia, M Rindisbacher. A Monte Carlo method for optimal portfolios. Journal of Finance, 2003, 58(1): 401-446.

[4] E Gobet, J Lemor, X Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. Annals of Applied Probability, 2005, 15(3): 2172-2202.

[5] H Markowitz. Portfolio selection. Journal of Finance, 1952, 7(1): 77-91.

[6] PA Samuelson. Lifetime portfolio selection by dynamic stochastic programming. Review of Economics and Statistics, 1969, 51(3):239-246.

[7] RC Merton. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory, 1971, 3(4): 373-413.

[8] SR Pliska. A stochastic calculus model of continuous trading: Optimal portfolios. Mathematics of Operations Research, 1986, 11(2): 371-383.

[9] I Karatzas, JP Lehoczky, SE Shreve. Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. SIAM Journal on Control and Optimization, 1987, 25(6): 1557-1586.

[10] JC Cox, C Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory, 1989, 49(1):33-83.

[11] L Hansen, T. Sargent. Robust control and model uncertainty. American Eco- nomic Review, 2001, 91(2): 60-66.

[12] RC Merton. On estimating the expected return on the market: An exploratory investigation. Journal of Financial Economics, 1980, 8(4): 323-361.

[13] I Gilboa, D Schmeidler. Max-min expected utility with non-unique priors. Journal of Mathematical Economics, 1989, 18(2): 141-153.

[14] L Epstein, T Wang. Intertemporal asset pricing under Knightian uncertainty. Econometrica, 1994, 62(2): 283-322.

[15] L Epstein, M. Schneider. Recursive multiple priors. Journal of Economic Theory, 2003, 113: 1-31.

[16] L Epstein, M Schneider. Learning under ambiguity. Review of Economic Study, 2007, 74(4)1275-1303.

[17] L Epstein, M Schneider. Ambiguity, information quality and asset pricing. Journal of Finance, 2008, 63(1): 197-228.

[18] P Klibanoff, M Marinacci, S Mukerji. A smooth model of decision making under ambiguity. Econometrica, 2005, 73(6): 1849-1892.

[19] H Chen, N Ju, J Miao. Dynamic asset allocation with ambiguous return predictability. AFA 2010 Atlanta Meeting Paper, 2011.

[20] N Ju, J Miao. Ambiguity, learning, and asset returns. Econometrica, 2012, 80(2): 559-591.

[21] EW Anderson, LP Hansen, TJ Sargent. A quartet of semigroups for model specification, robustness, prices of risk, and model detection. Journal of the Euro- pean Economic Association, 2003, 1(1): 68-123.

[22] C Skiadas. Smooth ambiguity aversion toward small risks and continuous-time recursive utility. Journal of Political Economy, 2013, 121(4): 775-792.

[23] P Maenhout. Robust portfolio rules and asset pricing. Review of Financial Studies, 2004, 17(4): 951-983.

[24] C Skiadas. Robust control and recursive utility. Finance and Stochastics, 2003, 7(4): 475-489.
[25] DJ Meyer, J Meyer. Measuring Risk Aversion. Foundations and trends in microeconomics. Now Publishers, 2006.
[26] JY Campbell, YL Chan, LM Viceira. A multivariate model of strategic asset allocation. Journal of Financial Economics, 2003, 67(1): 41-80.
[27] F Maccheroni, M Marinacci, A Rustichini. Ambiguity aversion, malevolent nature, and the variational representation of preferences. Econometrica, 2006, 74(6): 1447-1498.
[28] J Miao, T Hayashi. Intertemporal substitution and recursive smooth ambiguity preferences. Economics, 2011, 6(3): 423-472.
[29] P Maenhout. Robust portfolio rules and detection-error probabilities for a mean- reverting risk premium. Journal of Economic Theory, 2006, 128(1): 136-163.
[30] J Berger. Statistical Decision Theory and Bayesian Analysis. Springer, 2nd edition, 1985.
[31] D Nualart. The Malliavin Calculus and Related Topics. Springer Verlag, 1995.
[32] NE Karoui, S Peng, MC Quenez. Backward stochastic differential equations in finance. Mathematical Finance, 1997, 7(1): 1-71.
[33] LP Hansen. Large sample properties of generalized method of moments estimators. Econometrica, 1982, 50(4): 1029-1054.
[34] LP Hansen, TJ Sargent. Robustness. Princeton University Press, 2008.
[35] AR Pedersen. A new approach to maximum likelihood estimation for stochastic differential equations based on discrete observations. Scandinavian Journal of Statistics, 1995, 22(1): 55-71.
[36] C Skiadas. Recursive utility and preferences for information. Economic Theory, 1998, 12: 293-312.

APPENDIX A

Before the proof of the main result, Theorem 3.1, we present the following results.

Lemma 1. The process V_t^x defined in Equation (2.2) can be expressed as:

$$V_t^x = \widetilde{E}_t\left[\int_t^T \exp\left(\int_t^s - \beta_v dv)\right) u(c_s) \exp\left(\frac{\eta}{2(\eta-1)^2}\int_t^s x_v^{'} x_v dv\right) ds\right]. \qquad (A.1) \tag{46}$$

Proof of Lemma 1.

$$V_t^x = E_t\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right) u(c_s)\left(d_{t,s}^x\right)^\kappa ds\right] = \widetilde{E}_t\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right) u(c_s) \exp\left(\frac{\kappa^2}{2}\int_t^s x_v^{'} x_v dv - \frac{\kappa}{2}\int_t^s x_v^{'} x_v dv\right) ds\right]. \tag{47}$$

The last equation gives:

$$V_t^x = \widetilde{E}_t\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right) u(c_s) \exp\left(\frac{\eta}{2(\eta-1)^2}\int_t^s x_v^{'} x_v dv\right) ds\right]. \tag{48}$$

Lemma 2. Define η_x as $\kappa^2 - \kappa = \frac{\eta}{(1-\eta)^2}$. There exists an adapted process σ_t^x such that the expression of V_t^x in Equation (A.1) has the dynamics:

$$dV_t^x = \left(\beta_t V_t^x - u_t - \frac{\eta_x}{2} x_t^{'} x_t V_t^x\right) dt + (\sigma_t^x)^{'} d\widetilde{B}_t, \qquad (A.2) \tag{49}$$

with the boundary condition $V_T^x = 0$.

Proof of Lemma 2. According to the PDE (A.2), we can express V_t^x as:

$$V_t^x = \exp\left(\int_0^t \beta_s ds\right) \exp\left(-\frac{\eta_x}{2}\int_0^t x_s^{'} x_s ds\right) f(t, c_t), \tag{50}$$

with the boundary condition $f(T, c_T) = 0$, for some function $f(\cdot,\cdot)$. By taking derivation on both sides of the equation above and equalizing it with the PDE (A.2), we have:

$$dV_t^x = \left(\beta_t V_t^x - u_t - \frac{\eta_x}{2} x_t^{'} x_t V_t^x\right) dt + (\sigma_t^x)' d\widetilde{B}_t. \tag{51}$$

As a result, we have $df(t, c_t)$ as:

$$df(t, c_t) = \exp\left(-\int_0^t \beta_s ds\right) \exp\left(\frac{\eta_x}{2}\int_0^t x_s^{'} x_s ds\right)\left(-u_t dt + (\sigma_t^x)' d\widetilde{B}_t\right). \tag{52}$$

Integrating from t to T and applying the boundary condition, we have:

$$V_t^x = \int_t^T \left(\exp\left(-\int_t^s \beta_v dv\right) \exp\left(\frac{\eta_x}{2}\int_t^s x_v^{'} x_v dv\right)\right)\left(u_s ds - (\sigma_s^x)^{'} d\widetilde{B}_s\right). \tag{53}$$

By taking expectation under $\widetilde{P}$ on both sides, we have:

$$V_t^x = \widetilde{E}_t\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right) \exp\left(\frac{\eta_x}{2}\int_t^s x_v^{'} x_v dv\right) u(c_s) ds\right]. \tag{54}$$

The existence of the adapted process σ_t^x is given by the martingale representation theorem.

From lemma 1, we can express V_t^x as:

$$V_t^x = \widetilde{E}_t\left[\int_t^T \exp\left(-\int_t^s \beta_v dv\right)\right) u(c_s) \exp\left(\frac{\eta_x}{2}\int_t^s x_v^{'} x_v dv\right) ds\right]. \tag{55}$$

Proof of Theorem 3.1. Denote the discounted version for a process V_t as:

$$\overline{V}_t = \exp\left(-\int_0^t \beta_s ds\right) V_t. \tag{56}$$

The dynamics of $\overline{V}_t^x$ are:

$$d\overline{V}_t^x = -\beta_t \exp\left(-\int_0^t \beta_s ds\right) V_t^x dt + \exp\left(-\int_0^t \beta_s ds\right) dV_t^x = -\left(\bar{u}_t + \frac{\eta_x}{2} x_t^{'} x_t \overline{V}_t^x\right) dt + (\overline{\sigma}_t^x)^{'} d\widetilde{B}_t. \tag{57}$$

The dynamics of V_t under the P measure is:

$$dV_t = -\left(u_t - \beta_t V_t - \frac{\eta}{2V_t}(\sigma_t^b)^{'} \sigma_t^b\right) dt + (\sigma_t^b)' dB_t. \qquad (A.3) \tag{58}$$

Convert the dynamics of V_t into the $\widetilde{P}$ measure:

$$dV_t = -\left(u_t - \beta_t V_t - \frac{\eta}{2V_t}(\sigma_t^b)^{'} \sigma_t^b - \kappa(\sigma_t^b)^{'} x_t\right) dt + (\sigma_t^b)^{'} d\widetilde{B}_t. \tag{59}$$

The dynamics of $d\overline{V}_t$ is:

$$d\overline{V}_t = -\left(\bar{u}_t - \frac{\eta}{2\overline{V}_{t_t}}(\overline{\sigma}_t^b)^{'} \overline{\sigma}_t^b - \kappa(\overline{\sigma}_t^b)^{'} x_t\right) dt + ((\overline{\sigma}_t^b)^{'} d\widetilde{B}_t. \tag{60}$$

Combining the dynamics of $d\overline{V}_t^x$ and $d\overline{V}_t$, we have:

$$d(\overline{V}_t^x - \overline{V}_t) = -\frac{\eta_x}{2} x_t^{'} x_t (\overline{V}_t^x - \overline{V}_t) dt + (\overline{\sigma}_t^x - \overline{\sigma}_t^b)^{'} d\widetilde{B}_t - \frac{1}{2}\left(x_t\sqrt{\eta_x \overline{V}_t} + \overline{\sigma}_t^b \sqrt{\frac{\eta}{\overline{V}_t}}\right)^{'} \left(x_t\sqrt{\eta_x \overline{V}_t} + \overline{\sigma}_t^b \sqrt{\frac{\eta}{\overline{V}_t}}\right) dt. \tag{61}$$

Note that in the dynamics of V_t in Equation (A.3), if $u > 0 (u < 0)$, then we have $V > 0 (V < 0)$ by Theorem A2 in Schroder and Skiadas (1999)[1]. The relationship holds under the $\widetilde{P}$ measure, as $\widetilde{P}$ is equivalent to P. As we specify $\eta < 0 (0 \leq \eta < 1)$ for the utility process $u < 0 (u > 0)$, we have $\frac{\eta}{V_t} \geq 0$.

Denote the process K_t as:

$$K_t = (\overline{V}_t^x - \overline{V}_t) \exp\left(\frac{\eta_x}{2}\int_0^t x_s^{'} x_s ds\right). \tag{62}$$

The dynamics of K_t are given by:

$$dK_t = \exp\left(\frac{\eta_x}{2}\int_0^t x_s^{'} x_s ds\right)\left(-\frac{1}{2}\left(x_t\sqrt{\eta_x \overline{V}_t} + \overline{\sigma}_t^b \sqrt{\frac{\eta}{\overline{V}_t}}\right)^{'}\left(x_t\sqrt{\eta_x \overline{V}_t} + \overline{\sigma}_t^b \sqrt{\frac{\eta}{\overline{V}_t}}\right) dt + (\overline{\sigma}_t^x - \overline{\sigma}_t^b)^{'} d\widetilde{B}_t\right). \tag{63}$$

Integrate on both sides from t to T and take expectation under $\widetilde{P}$:

$$K_T - K_t = \widetilde{E}_t\left[\int_t^T \exp\left(\frac{\eta_x}{2}\int_0^s x_v^{'} x_v dv\right)\left(-\frac{1}{2}\left(x_t\sqrt{\eta_x \overline{V}_s} + \overline{\sigma}_s^b \sqrt{\frac{\eta}{\overline{V}_s}}\right)^{'}\left(x_s\sqrt{\eta_x \overline{V}_s} + \overline{\sigma}_s^b \sqrt{\frac{\eta}{\overline{V}_s}}\right) ds\right)\right]. \tag{64}$$

Replace K_t with $(\overline{V}_t^x - \overline{V}_t)\exp\left(\frac{\eta_x}{2}\int_0^t x_s^{'} x_s ds\right)$ and apply boundary conditions on $\overline{V}_t$ and $\overline{V}_t^x$:

$$\overline{V}_t^x - \overline{V}_t = \widetilde{E}_t\left[\int_t^T \exp\left(\frac{\eta_x}{2}\int_0^s x_v^{'} x_v dv\right)\left(\frac{1}{2}\left(x_t\sqrt{\eta_x \overline{V}_s} + \overline{\sigma}_s^b \sqrt{\frac{\eta}{\overline{V}_s}}\right)^{'}\left(x_s\sqrt{\eta_x \overline{V}_s} + \overline{\sigma}_s^b \sqrt{\frac{\eta}{\overline{V}_s}}\right) ds\right)\right]. \tag{65}$$

Convert the discount back, we get :

$$V_t^x - V_t = \widetilde{E}_t\left[\int_t^T \exp\left(\frac{\eta_x}{2}\int_0^s x_v^{'} x_v dv\right)\left(\frac{1}{2}\left(x_t\sqrt{\eta_x V_s} + \sigma_s^b \sqrt{\frac{\eta}{V_s}}\right)^{'}\left(x_s\sqrt{\eta_x V_s} + \sigma_s^b \sqrt{\frac{\eta}{V_s}}\right) ds\right)\right]. \tag{66}$$

The term on the right-hand side is greater or equal to zero, with the zero-value obtained by imposing below:

$$x_t = -\sigma_t^b \left|\frac{1-\eta}{V_t}\right|. \tag{67}$$

Proof of Proposition 3.2. First consider the case $V_t > 0$. The ordinally equivalent transformation of V_t is

$$v_t = V_t^{1-\eta}. \tag{68}$$

By the Ito's lemma, we have:

$$dv_t = (1-\eta) V_t^{-\eta}\left(-(u_t - \beta_t V_t) dt + (\sigma_t^b)^{'} dB_t\right). \tag{69}$$

Integrate and take expectations on both sides:

$$v_T - v_t = E_t\left[\int_t^T (1-\eta)V_s^{-\eta}(-u_s + \beta_s V_s)ds\right]. \tag{70}$$

Replace V_t with $v_t^{\frac{1}{1-\eta}}$, we have:

$$v_t = E_t\left[\int_t^T (1-\eta)\left(v_s^{\frac{\eta}{\eta-1}}u_s - \beta_s v_s\right)ds\right]. \tag{71}$$

The similar analysis can be applied to $V_t < 0$ gives:

$$v_t = E_t\left[\int_t^T (1-\eta)\left((-v_s)^{\frac{\eta}{\eta-1}}u_s - \beta_s v_s\right)ds\right]. \tag{72}$$

By taking two cases together, we have:

$$v_t = E_t\left[\int_t^T (1-\eta)\left(|v_s|^{\frac{\eta}{\eta-1}}u_s - \beta_s v_s\right)ds\right]. \tag{73}$$

Proof of Theorem 4.1. The state variable Y_t has the dynamics:

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dB_t. \tag{74}$$

The Maliavin derivative D_sY_t follow the dynamics:

$$d(D_sY_t) = \partial_2\mu(t, Y_t)(D_sY_t)dt + \partial_2\sigma(t, Y_t)(D_sY_t)dB_t, \tag{75}$$

with initial condition $D_sY_s = \sigma(Y_s)$.

For the case of the CRRA utility, Schroder and Skiadas (1999) show that the pair (J, Z) follows the process[1]:

$$-dJ_t = F(t, \theta_t, \sigma_t, Y_t, Z_t) - Z_t^{'}dB_t, \tag{76}$$

where

$$F(t, \theta_t, \sigma_t, Y_t, Z_t) = \frac{1}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}} + \frac{1-\gamma}{\gamma}\left(r_t - \frac{\beta}{1-\gamma} + \frac{k\theta_t^{'}\theta_t}{2}\right)J_t + \frac{\alpha k}{2}\frac{Z_t^{'}Z_t}{J_t} - Z_t^{'}(1-k)\theta_t. \tag{77}$$

In this BSDE associated with a forward equation, for $0 \le s \le t \le T$, the dynamics of Malliavin derivative D_sY_t are:

$$-D_sJ_t = G(s, t, \theta_t, \sigma_t, Y_t, Z_t)dt - (dB_t)^{'}D_sZ_t, \tag{78}$$

with

$$\begin{aligned} &G(s, t, \theta_t, \sigma_t, Y_t, Z_t) \\ &= -(1-k)\left(\theta_t^{'}D_sZ_t + Z_t^{'}D_s\theta_t\right) + \frac{1-\gamma}{\gamma}\left(\left(D_sr_t + k\theta_t^{'}D_s\theta_t\right)J_t + \left(r_t - \frac{\beta}{1-\gamma} + \frac{k\theta_t^{'}\theta_t}{2}\right)D_sJ_t\right) \\ &+ \frac{\alpha k}{2}\left(\frac{2Z_t^{'}}{J_t}D_sZ_t - \frac{Z_t^{'}Z_t}{J_t^2}D_sJ_t\right), \end{aligned} \tag{79}$$

and the initial condition is:

$$D_sJ_T = 0 \times 1^{'}. \tag{80}$$

The calculation rule and smoothing conditions can be found in Karoui et al. (1997)[32].

Proof of Proposition 4.2. The dynamics of X_t is given by Schroder and Skiadas (1999)[1]:

$$dX_t = -\left[\frac{\alpha}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}}J_t^{-1} - (1+\alpha)\beta + r_t + \frac{\theta_t^{'}\theta_t}{2}\right]dt - \theta_t^{'}dB_t. \tag{81}$$

For $0 \le s \le t \le T$, the Malliavin derivative D_sX_t has the dynamics:

$$dD_sX_t = -\left[D_sr_t - \frac{\alpha}{1-\gamma}(1+\alpha)^{\frac{1-\gamma}{\gamma}}J_t^{-2}D_sJ_t\right]dt - (dB_t + \theta_t dt)^{'}D_s\theta_t, \tag{82}$$

with initial condition:

$$D_sX_s = -\theta_s^{'}. \tag{83}$$

Proof of Theorem 4.3. The optimal wealth w_t at time t is

$$\xi_t w_t = E_t\left[\int_t^T \xi_u c_u du\right]. \tag{84}$$

By the Ito's lemma, the diffusion process on the left-hand side of the equation above is:

$$-\xi_t w_t\theta_t^{'} + \xi_t w_t\pi_t^{'}\sigma_t. \tag{85}$$

By the Clark-Ocone formula, the diffusion process on the right-hand side of the equation above is:

$$E_t\left[\int_t^T D_t(\xi_s c_u)du\right]. \tag{86}$$

The chain rule of Malliavin calculus gives:

$$D_t(\xi_u c_u) = \xi_u D_t(c_u) - c_u\xi_u\left(\theta_t^{'} + H_{t,u}\right). \tag{87}$$

Then we have:

$$E_t\left[\int_t^T D_t(\xi_u c_u)du\right] = E_t\left[\int_t^T \left(\xi_u D_t(c_u) - c_u\xi_u H_{t,u}\right)du\right] - w_t\xi_t\theta_t^{'}. \tag{88}$$

Equate both sides gives:

$$\xi_t w_t \sigma_t^{'}\pi_t = E_t\left[\int_t^T \left(\xi_u D_t(c_u) - c_u\xi_u H_{t,u}\right)du\right]. \tag{89}$$

That is

$$w_t \sigma_t^{'}\pi_t = E_t\left[\int_t^T \left(\xi_{t,u} D_t(c_u) - c_u\xi_{t,u} H_{t,u}\right)du\right]. \tag{90}$$

Denote $X_{t,u}$ as $X_u - X_t$. Applying the chain rule of Malliavin calculus, we have:

$$D_t(c_u) = c_u\left(-kD_tX_{t,u} + \frac{\alpha k}{J_i}D_tJ_u\right) + k\theta_t^{'}c_t. \tag{91}$$

Rearrange the terms and we have:

$$w_t\pi_t^{'}\sigma_t = E_t\left[\int_t^T \xi_{t,u}c_u\left(-kD_tX_{t,u} + \frac{\alpha k}{J_u}D_tJ_u\right)ds\right] - E_t\left[\int_t^T c_u\xi_{t,u}H_{t,u}ds\right] + k\theta_t^{'}w_t. \tag{92}$$

Proof of Theorem 4.4. The dynamics of (Y_t, D_sY_t) are the same as in Theorem 4.1. For the case of the logarithmic utility, Schroder and Skiadas (1999) show that the process (J, Z) follows the process[1]:

$$-dJ_t = F(t,\theta_t,\sigma_t,Y_t,Z_t) - Z_t^{'}dB_t, \tag{93}$$

with

$$F(t,\theta_t,\sigma_t,Y_t,Z_t) = (1-k_t)\left(\beta - r_t - \frac{k_t\theta_t^{'}\theta_t}{2} - Z_t^{'}\theta_t\right) + k_t(\alpha-\beta)J_t + \frac{1}{2}Z_t^{'}Z_t. \tag{94}$$

In this BSDE associated with a forward equation, for $0 \le s \le t \le T$, the dynamics of the Malliavin derivative D_sY_t are

$$-D_sJ_t = G(s,t,\theta_t,\sigma_t,Y_t,Z_t)dt - (dB_t)^{'}D_sZ_t, \tag{95}$$

with

$$G(s,t,\theta_t,\sigma_t,Y_t,Z_t) = k_t(\alpha-\beta)D_sJ_t + Z_t^{'}D_sZ_t + (1-k_t)\left(-D_sr_t - k_t\theta_t^{'}D_s\theta_t - Z_t^{'}D_s\theta_t - \theta_t^{'}D_sZ_t\right), \tag{96}$$

and the initial condition is:

$$D_sJ_T = 0 \times 1^{'}. \tag{97}$$

Proof of Proposition 4.5. Express X_t as:

$$X_t = \exp\left(-\int_0^t \left((\beta_v - \alpha)k_v - \beta\right)dv\right)K_t. \tag{98}$$

The Ito's lemma and the dynamics of X_t gives:

$$\begin{aligned} dX_t &= -X_t\left((\beta-\alpha)k_t - \beta\right)dt + \exp\left(-\int_0^t \left((\beta-\alpha)k_v - \beta\right)dv\right)dK_t \\ &= -\left(\left((\beta-\alpha)k_t - \beta\right)X_t + (\alpha-\beta)J_t - \beta + r_t + \frac{\theta_t^{'}\theta_t}{2}\right)dt - \theta_t^{'}dB_t. \end{aligned} \tag{99}$$

As a result, we have:

$$dK_t = -\left(\left((\alpha-\beta)J_t - \beta + r_t + \frac{\theta_t^{'}\theta_t}{2}\right)dt + \theta_t^{'}dB_t\right)\exp\left(\int_0^t \left((\beta-\alpha)k_v - \beta\right)dv\right), \tag{100}$$

with

$$K_0 = \log(\lambda). \tag{101}$$

Integrate on K_t, and plug it back into the expression of X_t, we have:

$$\begin{aligned} X_t = &-\int_0^t \exp\left(-\int_s^t \left((\beta-\alpha)k_v - \beta\right)dv\right)\left(\left((\alpha-\beta)J_s - \beta + r_s + \frac{\theta_s^{'}\theta_s}{2}\right)ds + \theta_s^{'}dB_s\right) \\ &+ \exp\left(-\int_0^t \left((\beta-\alpha)k_v - \beta\right)dv\right)\log(\lambda). \end{aligned} \tag{102}$$

The chain rule of Malliavin calculus gives:

$$dD_sX_t = -\left(\left((\beta-\alpha)k_t - \beta\right)D_sX_t + (\alpha-\beta)D_sJ_t + D_sr_t + \theta_t^{'}D_s\theta_t\right)dt - (dB_t)^{'}D_s\theta_t, \tag{103}$$

with the initial condition:

$$D_sX_s = -\theta_s^{'}. \tag{104}$$

By the similar analysis for X_t, the process D_sX_t can be expressed as:

$$D_sX_t = -\int_s^t \exp\,(-\int_v^t \exp\left(-\int_v^t \left((\beta-\alpha)k_l-\beta\right)dl\right)\left(\left((\alpha-\beta)D_sJ_v + D_sr_v + \theta_v^{'}D_s\theta_v\right)dv + (dB_s)^{'}D_s\theta_v\right) + \exp\left(-\int_s^t \left((\beta-\alpha)k_l-\beta\right)dl\right)\theta_s^{'}. \quad (105)$$

Proof of Theorem 4.6. The expression of πt is the same as in Theorem 4.3:

$$w_t\pi_t^{'}\sigma_t = E_t\left[\int_t^T \left(\xi_{t,u}D_t(c_u) - c_u\xi_{t,u}H_{t,u}\right)du\right]. \quad (106)$$

By applying the chain rule of Malliavin calculus, we have:

$$D_t(c_u) = c_u(D_tJ_u - k_uD_tX_u) = c_u\left(D_tJ_u - k_uD_tX_{t,u} + k_u\theta_t\right). \quad (107)$$

Then the optimal portfolio can be expressed as:

$$w_t\pi_t^{'}\sigma_t = E_t\left[\int_t^T \left(D_tJ_u - k_uD_tX_{t,u}\right)c_u\xi_{t,u}du\right] - E_t\left[\int_t^T c_u\xi_{t,u}H_{t,u}du\right] + \theta_tE_t\left[\int_t^T c_u\xi_{t,u}k_udu\right]. \quad (108)$$

Replacing the expression of $D_tX_{t,u}$ of Proposition (4.5), we can express π_t as:

$$w_t\pi_t^{'}\sigma_t = E_t\left[\int_t^T \left(D_tJ_u + k_u\left(\int_t^u e^{\int_t^v ((\beta-\alpha)k_l-\beta)dl}\left((\alpha-\beta)D_tJ_v\right)dv\right)\right)c_u\xi_udu\right] - E_t\left[\int_t^T \left(\int_t^u \exp^{\int_v^t ((\beta-\alpha)k_l-\beta)dl}\left(D_tr_vdv + (dB_v+\theta_vdv)^{'}D_t\theta_v\right)dv\right)k_uc_u\xi_{t,u}du\right]. \quad (109)$$