Eurasia Journal of Science and Technology
ISSN: 2663-1016
DOI: https://doi.org/10.61784/ejst3085

THE DECOUPLING OF SELF-ADJOINT SECOND-ORDER SYSTEMS WITH STRUCTURE-PRESERVING TRANSFORMATION

Nan Jiang, QiZhi Zhang*
Northeast Forestry University, Harbin 150040, Heilongjiang, China.
Corresponding Author: QiZhi Zhang, Email: zqz_math@nefu.edu.cn

Abstract: This paper focuses on the decoupling of self-adjoint second-order linear systems. It is proposed a method with congruence transformation matrix that preserving the Lancaster structure. This method requires fewer parameters during the decoupling while remaining the spectrum of the system before and after decoupling. Numerical simulation experiments demonstrate the implementation results achieved by applying this method to system decoupling.
Keywords: Self-adjoint; Second-order systems; Decoupling; Lancaster structure

1 INTRODUCTION

Typically, dynamic systems inherently possess highly intricate structures. It is defined as a coupled system when the system consists of interconnected subsystems. Coupled systems find extensive applications across diverse domains such as aerospace technology, economic development, and agricultural production[1-3]. However, the complex configurations of coupled system devices and apparatuses, combined with interactions and mutual influences among all or partial subsystems, introduce numerous uncertain, complex, and difficult-to-control factors during system analysis. These challenges ultimately hinder the effective optimization of system performance[4-8]. Currently, to address the challenges in system analysis caused by mutual interference among subsystems in coupled systems, engineers primarily employ system decoupling methods. These approaches preserve the intrinsic properties of the original coupled system while eliminating all or partial coupling relationships, thereby transforming a multi-degree-of-freedom system into multiple independent single-degree-of-freedom subsystems without mutual interactions. Given that higher-order linear coupled systems can be effectively reduced to second-order linear coupled systems through appropriate treatment, this study focuses on the decoupling of self-adjoint second-order linear dynamical systems. We present a congruence transformation of decoupling with preserving the Lancaster structure, accompanied by a series of numerical simulation experiments.

2 TRANSFORMATION FOR SELF-ADJOINT SECOND-ORDER LINEAR SYSTEMS

The second-order linear dynamical systems is represented by differential equations: $M_0\ddot{\boldsymbol{x}}(t)+C_0\dot{\boldsymbol{x}}(t)+K_0\boldsymbol{x}(t)=\boldsymbol{f}(t)$, where M_0, C_0 and K_0 are respectively the given initial coefficient matrices of an n-degree-of-freedom system, $\boldsymbol{x}(t)$ and $\boldsymbol{f}(t)$ are vectors with n degrees of freedom[9]. Such systems are extensively utilized in across critical disciplines such as applied mechanics, acoustics, circuit simulation, structural mechanics, fluid mechanics, and microelectronics design.

To perform decoupling on the coefficients of the second-order linear dynamical system, we first denote the Lancaster structure composed of the coefficient matrices as

$$\mathrm{L}(\lambda)=\mathrm{L}(\lambda;M_0,C_0,K_0)=B_0\lambda+A_0 \tag{1}$$

where

$$B_0=\begin{bmatrix} C_0 & M_0 \\ M_0 & 0 \end{bmatrix},\ A_0=\begin{bmatrix} K_0 & 0 \\ 0 & -M_0 \end{bmatrix}$$

the parameter λ represents spectrum of this system. The parameterized governing equations of the system are formulated as follow

$$M(t)\ddot{\boldsymbol{x}}(t)+C(t)\dot{\boldsymbol{x}}(t)+K(t)\boldsymbol{x}(t)=\boldsymbol{f}(t) \tag{2}$$

Then, denote the Lancaster structure[10] corresponding to the parameterized system (2) is as

$$\mathrm{L}(\lambda;M(t),C(t),K(t))=B(t)\lambda+A(t) \tag{3}$$

where

$$B(t)=\begin{bmatrix} C(t) & M(t) \\ M(t) & 0 \end{bmatrix},\ A(t)=\begin{bmatrix} K(t) & 0 \\ 0 & -M(t) \end{bmatrix} \tag{4}$$

Since the spectrum is critically important for determining the motion displacement and vibrational states of second-order linear systems, therefore, the spectrum of the parameterized system remains invariant. Additionally, all coefficient matrices are required to be differentiable with respect to the time parameter λ.

There must exist a set of $2n \times 2n$ invertible parameter-dependent real transformation $\Pi_L(t)$ and $\Pi_R(t)$, that ensure

$$\begin{cases} B(t) = \Pi_L^{\mathrm{T}}(t) B_0 \Pi_R(t) \\ A(t) = \Pi_L^{\mathrm{T}}(t) A_0 \Pi_R(t) \end{cases} \tag{5}$$

By differentiating both sides of Equations (5) with respect to the parameter λ, we obtain

$$\begin{cases} \dot{B}(t) = L^{\mathrm{T}}(t) B(t) + B(t) R(t) \\ \dot{A}(t) = L^{\mathrm{T}}(t) A(t) + A(t) R(t) \end{cases} \tag{6}$$

$L(t)$ and $R(t)$ are the introduced condition matrix. By reorganizing the aforementioned transformations, we derive the following system of equations comprising $5n^2$ equations with $8n^2$ unknowns

$$\begin{aligned} &L_{12}^{\mathrm{T}}(t) M(t) + M(t) R_{12}(t) = 0 \\ &K(t) R_{12}(t) - L_{21}^{\mathrm{T}}(t) M(t) = 0 \\ &L_{12}^{\mathrm{T}}(t) K(t) - M(t) R_{21}(t) = 0 \\ &L_{11}^{\mathrm{T}}(t) M(t) - L_{22}^{\mathrm{T}}(t) M(t) + C(t) R_{12}(t) = 0 \\ &M(t) R_{11}(t) - M(t) R_{22}(t) + L_{12}^{\mathrm{T}}(t) C(t) = 0 \end{aligned} \tag{7}$$

A second-order linear system is defined as self-adjoint when all its coefficient matrices are symmetric. When the initial coefficient matrices are all symmetric matrices, we should require preserving the symmetry of the coefficient matrices during decoupling. Specifically, matrices $M(t)$, $C(t)$ and $K(t)$ must retain their symmetric properties for any time parameter t. Introducing three $n \times n$ parameter-dependent matrices $D(t)$, $N_L(t)$ and $N_R(t)$, where $D(t)$ is a skew-symmetric matrix. This ensures that the symmetry of all coefficient matrices is preserved. We let $N_L^{\mathrm{T}}(t) = N_R(t) = N(t)$, $\Pi_L(t) = \Pi_R(t) = T(t)$. At this stage, the number of parameter matrices is reduced to two, and the number of independent parameters is decreased to $2n^2$.

Impose a constraint on the self-adjoint second-order linear system by normalizing the initial mass matrix M_0 to the identity matrix. Simultaneously, to preserve the self-adjoint property, the following constraint is imposed $M(t) \equiv I$, i.e. $\dot{M}(t) = 0$. Consequently, the system (7) can be simplified to

$$\begin{aligned} &K(t) R_{12}(t) - R_{21}^{\mathrm{T}}(t) M(t) = 0 \\ &R_{12}^{\mathrm{T}}(t) M(t) + M(t) R_{12}(t) = 0 \\ &M(t) R_{11}(t) - M(t) R_{22}(t) + R_{12}^{\mathrm{T}}(t) C(t) = 0 \end{aligned} \tag{8}$$

where the coefficient matrix is

$$R(t) = \begin{bmatrix} D(t) & 0 \\ 0 & D(t) \end{bmatrix} \begin{bmatrix} -\dfrac{C(t)}{2} & -M(t) \\ K(t) & \dfrac{C(t)}{2} \end{bmatrix} + \begin{bmatrix} N(t) & 0 \\ 0 & N(t) \end{bmatrix} \tag{9}$$

Now the number of unknowns is reduced to $n(n-1)/2 + n^2$. To further simplify the number of parameters, a skew-symmetric matrix is introduced that satisfied the following condition

$$N(t) - N^{\mathrm{T}}(t) = 2S(t) \tag{10}$$

Matrix $S(t)$ serves as the new parameter matrix capable of replacing the original parameter matrix $N(t)$. After substitution, matrix $N(t)$ can be expressed as a combination of two skew-symmetric matrices $D(t)$ and $S(t)$

$$N(t) = \frac{1}{4}\big(C(t)D(t) - D(t)C(t)\big) + S(t) \tag{11}$$

Since the parameter matrices are now both skew-symmetric, the number of parameters has been simplified to $n(n-1)$. Furthermore, the derivative equation of the congruence transformation matrix $T(t)$ with respect to time t is derived as follows

$$\dot{T}=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} \frac{1}{4}CD-\frac{3}{4}DC+S & -D \\ DK & \frac{1}{4}CD+\frac{1}{4}DC+S \end{bmatrix} \tag{12}$$

Herein, the description of parameter t is omitted, with $T(t)$ defined as the derivative of matrix $\dot{T}$ with respect to time parameter t. By appropriately selecting only $n(n-1)$ parameters from the skew-symmetric matrices $D(t)$ and $S(t)$, the transformation $T(t)$ can be obtained. Subsequently, a congruence transformation is applied to the Lancaster structure using the transformation $T(t)$. Through suitable optimal control methods, the trajectories of matrices $M(t)$, $C(t)$ and $K(t)$ are tracked at different time instants t, thereby driving these matrices progressively toward diagonal structures. A self-adjoint second-order linear system is said to achieve complete decoupling, at a specific time instant t, all transformed coefficient matrices are diagonal matrices.

3 NUMERICAL SIMULATION EXPERIMENTS

A numerical simulation verification is provided for the structure-preserving congruence transformation solving algorithm targeting self-adjoint second-order linear systems. The experiments were implemented using MATLAB programming, where the iterative integration method invokes MATLAB's standard ODE integrators. Both the absolute error tolerance (AbsTol) and relative error tolerance (RelTol) in the simulation program were set to 10^{-10}, with the output data retaining four significant decimal places.

Given a self-adjoint second-order linear system with 4 degrees of freedom, its initial mass matrix $M_0=I$, initial damping C_0 and stiffness matrices K_0 are defined as follows

$$C_0=\begin{bmatrix} 0.4108 & 0.0000 & -0.3529 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.3529 & 0.0000 & 2.1661 & -1.8132 \\ 0.0000 & 0.0000 & -1.8132 & 1.8132 \end{bmatrix}$$

$$K_0=\begin{bmatrix} 22.3480 & -9.3547 & -8.9365 & 0.0000 \\ -9.3547 & 18.5240 & -9.1690 & 0.0000 \\ -8.9365 & -9.1690 & 22.2080 & -4.1027 \\ 0.0000 & 0.0000 & -4.1027 & 4.1027 \end{bmatrix} \tag{13}$$

By employing the transformation $T(t)$ solving algorithm that maintains the Lancaster structure, appropriate numerical integration iterations are performed on the time parameter t. The integration process terminates at $t \approx 4.0\times 10^2$, and the damping matrix and stiffness matrix of the decoupled system are output as follows

$$C=\begin{bmatrix} 0.2073 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 2.2029 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.9436 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0363 \end{bmatrix}$$

$$K=\begin{bmatrix} 29.5482 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 6.0921 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 27.5510 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.8435 \end{bmatrix} \tag{14}$$

In this case, the decoupled mass matrix M is the identity matrix. As evidenced by the output data, the damping matrix C and stiffness matrix K are perfectly diagonal matrices. The total decoupled system represents four independent single-degree-of-freedom subsystems, where the damping coefficient and stiffness coefficient of each subsystem are given by the diagonal elements of matrices C and K, respectively.

The variation in the sum of off-diagonal elements for matrices C and K during the complete system decoupling process is graphically illustrated in Figure 1.

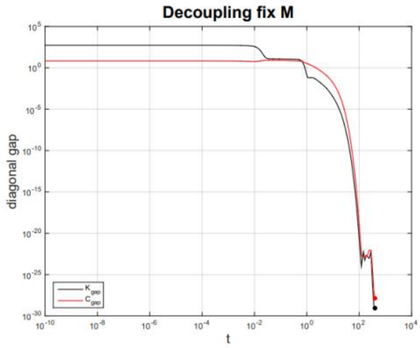

Figure 1 The Evolution of Off-Diagonal Portions of Matrices C and K

When the integration begins, the curves representing the sum of off-diagonal elements of the damping matrix and stiffness matrix start to descend at identical rates. Upon termination of the integration, both curves stabilize near the order of magnitude of 10^{-30}. Since the error tolerance in the program was set to 10^{-10}, this confirms that the system has been totally decoupled.

To validate spectrum invariance, the spectrum of the quadratic eigenvalue problem (QEP) associated with the total decoupled system is computed. The absolute error between the spectrum at each time parameter t and the initial spectrum is selected as the metric for spectral variation. The evolution of spectral variation during the decoupling is graphically illustrated in Figure 2.

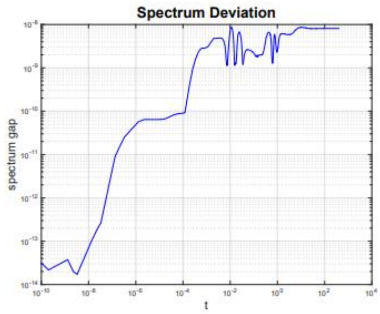

Figure 2 The Deviations of Spectrum

As illustrated in Figure 2, the spectrum of the totally decoupled system has changed. When the integration commences, spectral variations emerge, with their magnitude progressively increasing as the integration proceeds. Upon integration termination, the magnitude of spectral variations stabilizes near the order of 10^{-8}. Since the error tolerance in the numerical simulation was set to 10^{-10}, these spectral variations can be considered negligible.

4 CONCLUSION

Decoupling of second-order linear systems constitutes a pivotal challenge in system analysis. This paper focuses on self -adjoint second-order linear systems. We proposed a structure-preserving congruence transformation solving this problem. Leveraging the isospectral theory of Lancaster structure preservation for quadratic eigenvalue problems (QEPs) inherent to such systems, the algorithm constructs a derivative equation of the congruence transformation matrix with respect to the time parameter. Through appropriate numerical integration of this derivative equation, a congruence transformation matrix is obtained at a specific time, which nullifies the sum of non-prespecified off-diagonal elements in the coefficient matrices. Subsequent application of this matrix to transform the coefficient matrices achieves complete decoupling of the self-adjoint second-order linear system.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

FUNDING

This work is supported by the Fundamental Research Funds for the Central Universities (Grant No. 2572021BC08).

REFERENCES

[1] Zhang L P, Yang Z D, Qu Z Y, et al. Modeling and vibration decoupling control of multi-axial shaking table. Journal of Southwest Petroleum University, 2015, 409: 1878-1883.

[2] Zheng Y, Zhou Z C, Huang H. A multi-frequency MIMO control method for the 6DOF micro-vibration exciting system. Acta Astronautica, 2020, 170: 552-569.
[3] Madden B, Florin N, Mohr S, et al. Using the waste Kuznet's curve to explore regional variation in the decoupling of waste generation and socioeconomic indicators. Resources Conservation and Recycling, 2019, 149: 674-686.
[4] Chen W, Zhao H B, Li J F, et al. Land use transitions and the associated impacts on ecosystem services in the Middle Reaches of the Yangtze River Economic Belt in China based on the geo-informatic Tupu method. The Science of the Total Environment, 2020, 701: 134690. DOI:10.1016/j.scitotenv.2019.134690.
[5] Ravneet K, Omkar J, Rodney W E. The ecological and economic determinants of eastern redcedar (Juniperus virginiana) encroachment in grassland and forested ecosystems: A case study from Oklahoma. Journal of Environmental Management, 2020, 254: 109815. DOI:10.1016/j.jenvman.2019.109815.
[6] Sanye-Mengual E, Secchi M, Corrado S, et al. Assessing the decoupling of economic growth from environmental impacts in the European Union: A consumption-based approach. Journal of Cleaner Production, 2019, 236: 117535. DOI:10.1016/j.jclepro. 2019.07.010.
[7] Kumar S, Mahulikar S P. Aero-thermal analysis of lifting body configurations in hypersonic flow. Acta Astronautica, 2016, 126: 382-394.
[8] Li S B, Wang Z G, Barakos G N, et al. Research on the drag reduction performance induced by the counterflowing jet for waverider with variable blunt radii. Acta Astronautica, 2016, 127: 120-130.
[9] Li Anming. Modeling and simulation of dynamic systems. Beijing: National Defense Industry Press, 2012: 29-53.
[10] Lancaster P. Linearization of regular matrix polynomials. The Electronic Journal of Linear Algebra, 2008, 17: 21-27.