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SOME PROPERTIES OF HYPERBOLIC TRIGONOMETRIC FUNCTIONS

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Abstract: This paper investigates some fundamental properties of hyperbolic trigonometric functions on the hyperbolic number plane. Hyperbolic numbers form a commutative ring with zero divisors, generated by two real numbers via the Cartesian basis. Within this algebraic framework, this paper systematically establish the system of hyperbolic trigonometric functions. By fully leveraging the decomposition properties of hyperbolic numbers and their associated trigonometric functions, this work effectively overcomes the mathematical difficulties arising from the presence of zero divisors in the hyperbolic number ring. On this basis, this paper for the first time in the context of a hyperbolic number ring containing zero divisors, systematically derived and rigorously proved the addition theorems for the hyperbolic sine, cosine, tangent, and cotangent functions. On this basis, we have successfully derived and rigorously proved the addition theorems for the hyperbolic sine, cosine, tangent, and cotangent functions, establishing a complete system of angle addition formulas and laying a solid foundation for the theory of hyperbolic functions. The addition theorems established in this research will provide an important theoretical foundation for the further development of hyperbolic analysis in function theory, while simultaneously injecting new research momentum into the further study of the properties and expansions of hyperbolic trigonometric series. These theoretical achievements are expected to play significant roles in the study of hyperbolic differential equations, geometric analysis, and related physical problems.
Keywords: Hyperbolic numbers; Hyperbolic trigonometric functions; Identity transformation; Addition theorems

1 INTRODUCTION

Hyperbolic numbers, as a two-dimensional extension of the real number field, provide an algebraic framework with unique properties such as zero divisors. Ferhat Kuruz and Ali Dagdeviren studied matrices with hyperbolic number entries, establishing foundational properties of hyperbolic matrices [1]. Iskender Ozturk and Mustafa Ozdemir systematically investigated affine transformations in the hyperbolic number plane [2], revealing their invariants and geometric structure. Cayo D. and Nara P. applied hyperbolic geometry to surface determination via short curves [3], while Rachid A. et al. designed capstone courses for mathematics majors using complex and hyperbolic numbers [4]. Yuksel S. explored dual hyperbolic generalized Fibonacci numbers, further enriching the algebraic theory of hyperbolic systems [5].

In the complex domain, trigonometric functions play a central role in analysis, and analogous developments are sought in the hyperbolic setting. Chunli Li and Wenchang Chu computed several classes of definite integrals involving hyperbolic and trigonometric functions [6], providing precise analytical expressions. Medvegyev P. revisited the construction of elementary trigonometric functions [7], offering insights into their foundational structure. Youssef A. and Elhoucien E. derived cosine and sine addition laws with an automorphism [8], extending classical identities to broader algebraic contexts. Hassan A. and Mehdi D. proposed diffusive representations for fractional derivatives using sine and cosine functions [9], linking functional equations to fractional calculus. Ebanks B. studied the cosine-sine functional equation on semigroups [10], establishing general solutions under algebraic constraints.

Despite these advances, a systematic theory of hyperbolic trigonometric functions—particularly their addition theorems—remains underdeveloped. Guarin Garcia Julian derived transformations between trigonometric and hyperbolic functions based on physical models of cable statics [11]. Alibrahim Hamzah Ali and Das Saptarshi bridged p-special functions between generalized hyperbolic and trigonometric families [12], obtaining several connecting identities. Chunli Li and Wenchang Chu further evaluated improper integrals involving powers of inverse trigonometric and hyperbolic functions [13]. Stojiljkovic V. et al. established sharp bounds for trigonometric and hyperbolic functions with applications to fractional calculus [14]. Bagul J. et al. derived polynomial-exponential bounds for certain trigonometric and hyperbolic functions [15], contributing to quantitative estimates in analysis. However, a rigorous and systematic treatment of addition formulas for hyperbolic trigonometric functions within the hyperbolic number plane is still lacking, which motivates the present work.

2 THE BASIC THEORY OF HYPERBOLIC NUMBERS

Hyperbolic numbers are an extension of the real numbers, defined as numbers of the form:

$$\mathbb{H}=\mathbb{R}[h]=\{x_0+x_1h;\ x_0,\ x_1\in\mathbb{R},\ h^2=1,\ h\neq\mathbb{R}\}. \tag{1}$$

Let any two hyperbolic numbers be denoted as $h_1 = x_0 + x_1 h$ and $h_2 = y_0 + y_1 h$ (where $x_0, x_1, y_0, y_1 \in \mathbb{R}$, and $h^2 = 1,\ h \notin \mathbb{R}$), their definitions of equality, addition, and multiplication are as follows:

$$h_1 = h_2 \Leftrightarrow x_0 = y_0 \text{ and } x_1 = y_1, \tag{2}$$

$$h_1 + h_2 = (x_0 + y_0) + (x_1 + y_1)h, \tag{3}$$

$$h_1 h_2 = (x_0 y_0 + x_1 y_1) + (x_1 y_0 + x_0 y_1)h. \tag{4}$$

To simplify the power operation and function expansion of hyperbolic numbers, a special representation form called the idempotent basis can be introduced. Define the idempotent basis elements as: $h_+ = \frac{1}{2}(1+h)$ and $h_- = \frac{1}{2}(1-h)$. Meanwhile, for any hyperbolic number $h = x_0 + x_1 h$, difine $x_+ = x_0 + x_1$ and $x_- = x_0 - x_1$. At this point, the hyperbolic number can be rewritten as a linear combination of the idempotent basis:

$$h = x_+ h_+ + x_- h_-. \tag{5}$$

The idempotent basis has the following three key properties:

$$h = x_+ h_+ + x_- h_-, \tag{6}$$

$$h_+ h_- = 0. \tag{7}$$

The basis elements $\{h_+, h_-\}$ form another set of basis for $\mathbb{H}$, and every hyperbolic number has the unique idempotent representation.

Now given a hyperbolic number: $\delta = x_0 + x_1 h$, on the hyperbolic plane, the partial order is defined as follows it can be transformed via a coordinate change into: $\delta = \alpha h_+ + \beta h_-$. And $\alpha = x_0 + x_1,\ \beta = x_0 - x_1$.

For any hyperbolic number delta, the following holds:

$$\delta = \alpha h_+ + \beta h_-,\ \ \alpha, \beta \in \mathbb{R}, \tag{8}$$

$$\delta_1 \preceq \delta_2 \Leftrightarrow \delta_2 - \delta_1 \succeq 0, \tag{9}$$

$$\begin{cases} \delta_1 = \alpha_1 h_+ + \beta_1 h_- \\ \delta_2 = \alpha_2 h_+ + \beta_2 h_- \end{cases} \Rightarrow (\alpha_2 - \alpha_1)h_+ + (\beta_2 - \beta_1)h_- \pm 0 \Leftrightarrow \begin{cases} \alpha_2 \geq \alpha_1 \\ \beta_2 \geq \beta_1 \end{cases}, \tag{10}$$

$$\delta_1 \preceq \delta_2 \Leftrightarrow \delta_2 - \delta_1 \succeq 0 \Leftrightarrow \delta_1 - \delta_2 \in \mathbb{H}^+, \tag{11}$$

$$\mathbb{H}^+ = \{\delta = \alpha h_+ + \beta h_-; \alpha \geq 0, \beta \geq 0\}. \tag{12}$$

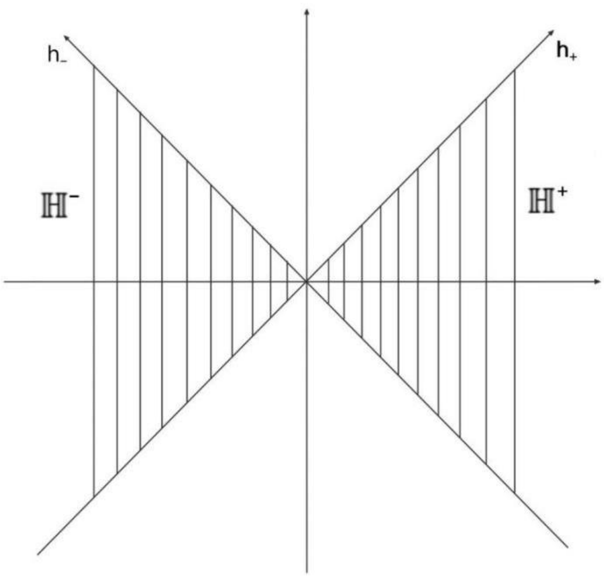

Figure 1 The Positive And Negative Hyperbolic Numbers

In Figure 1 illustrates the fundamental partition of the hyperbolic number plane into four sectors based on the signs of the idempotent components h_+ and h_-. The first and third quadrants, which are shaded, represent the sets of positive and negative hyperbolic numbers, respectively. This structure is fundamental to understanding the partial order and the behavior of functions defined on hyperbolic numbers, as it directly stems from the existence of zero divisors in the algebra.

Define a hyperbolic interval as follows:

$$[\delta,\omega]_{\mathbb{H}} = \{\xi \in \mathbb{H}: \delta \preceq \xi \preceq \omega\}. \tag{13}$$

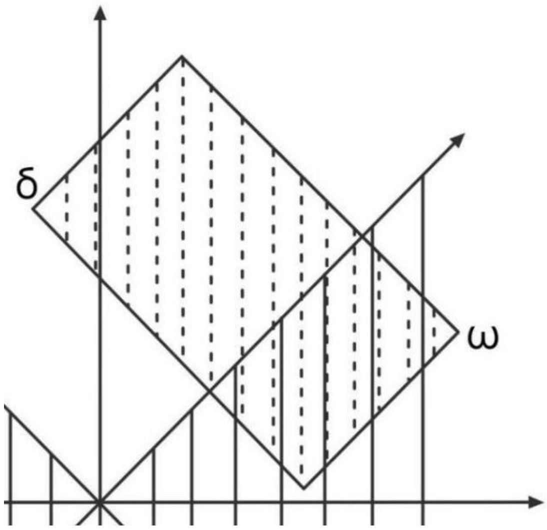

Figure 2 The Hyperbolic Interval $[\delta,\omega]_{\mathbb{H}}$.

In Figure 2 depicts a hyperbolic interval on the hyperbolic number plane. Geometrically, this interval represents a rectangular region defined by the partial order inherent to hyperbolic numbers, bounded by the idempotent components δ and ω. This structure generalizes the concept of a real-number interval to two dimensions within the hyperbolic plane. The hyperbolic interval is fundamental for defining domains of functions and convergence regions in hyperbolic analysis, providing a crucial geometric framework for subsequent theoretical developments.

The modulus of a hyperbolic number is defined as follows:

$$|\delta|_{\mathbb{H}} = |\alpha h_+ + \beta h_-|_{\mathbb{H}} = |\alpha| h_+ + |\beta| h_- \in \mathbb{H}^+. \tag{14}$$

3 DEFINITION AND PROPERTIES OF HYPERBOLIC TRIGONOMETRIC FUNCTIONS

3.1 Definition of Hyperbolic Trigonometric Functions

The computation of powers for hyperbolic numbers can be efficiently performed using idempotent bases. Specifically, for every $h \in \mathbb{H}$ and $n \in \mathbb{R}$, the following relation can be obtained according to formula (6) and formula (7):

$$h^n = (x_+h_+ + x_-h_-)^n = (x_+)^n (h_+)^n + (x_-)^n (h_-)^n = (x_+)^n h_+ + (x_-)^n h_-, \tag{15}$$

Formula (15) provides the idempotent basis representation for the power computation of hyperbolic numbers, laying the foundation for subsequent function definitions.

Similar to zero divisor factorization, if a hyperbolic function f is differentiable on the hyperbolic plane, then f admits a zero divisor factorization. Moreover, this operation can be formally expressed as a functional:

$$f(h) = f(x_+h_+ + x_-h_-) = f(x_+)h_+ + f(x_-)h_-. \tag{16}$$

Equation (16) gives the idempotent decomposition form of a function acting on a hyperbolic number, which transforms the functional operation on the domain of hyperbolic numbers into the functional operations on its two idempotent components.

Example 1. If $f(h) = \delta = \sin(h)$ and $h = x_0 + x_1 h$ then, It has:

$$f(h) = \delta = \sin(h) = \sin(x_0 + x_1 h) = \sin(x_+h_+ + x_-h_-) = \sin(x_+)h_+ + \sin(x_-)h_-, \tag{17}$$

Taking the hyperbolic interval $[-\pi,\pi]_{\mathbb{H}}$ as the domain, The graph of the hyperbolic trigonometric function is shown below:

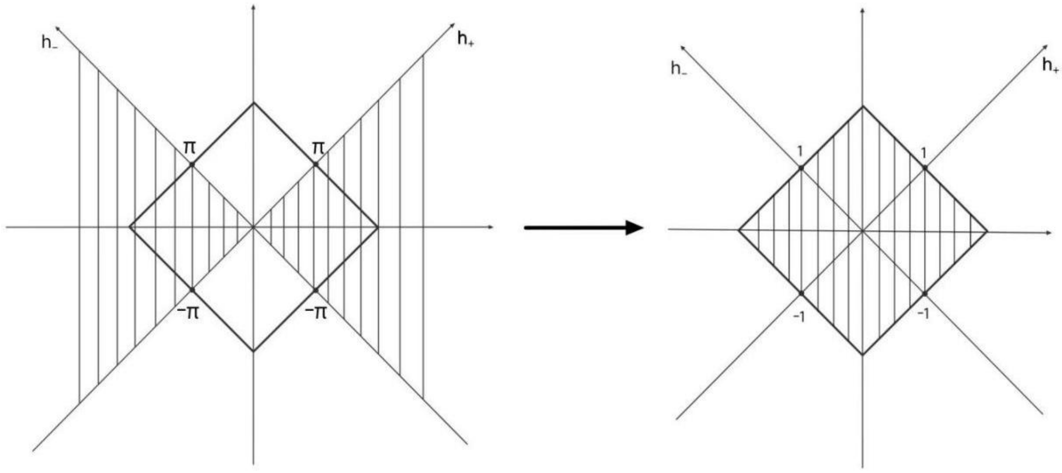

Figure 3 The Graph of The Hyperbolic Trigonometric Function

In Figure 3 presents the four-dimensional hyperbolic sine function through two two-dimensional cross-sectional views, illustrating the graph of the hyperbolic trigonometric function $f(h)$ over the hyperbolic interval $[-\pi,\pi]_{\mathbb{H}}$. These plots depict the variation patterns of the hyperbolic sine function within the hyperbolic number plane, revealing its unique oscillatory and growth characteristics. Geometrically, the visualization demonstrates how these functions evolve along the idempotent components, highlighting the intrinsic interplay between the algebraic structure of hyperbolic numbers and their geometric representation.

Let $\delta = x_0 + x_1 h = \alpha h_+ + \beta h_- \in \mathbb{H}$, then it follows that:

$$\begin{aligned} \sin(\delta) &= \sin(\alpha h_+ + \beta h_-) = \sin(\alpha) h_+ + \sin(\beta) h_-, \\ \cos(\delta) &= \cos(\alpha h_+ + \beta h_-) = \cos(\alpha) h_+ + \cos(\beta) h_-. \end{aligned} \tag{18}$$

By analogy with the complex domain, the hyperbolic tangent and cotangent functions are defined as follows:

If $\delta \in \mathbb{H}$, the hyperbolic tangent function is defined if and only if $\cos\delta$ is invertible, which requires that neither of values α and β equals $\frac{\pi}{2} + k\pi$ for any integer k . By computation, it follows that:

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \tan(\alpha) h_+ + \tan(\beta) h_-. \tag{19}$$

Similarly, if $\delta \in \mathbb{H}$, the hyperbolic cotangent function is defined if and only if $\sin\delta$ is invertible, which requires that neither of values α and β equals $k\pi$ for any integer k . By computation, it follows that

$$\cot\delta = \frac{\cos\delta}{\sin\delta} = \cot(\alpha) h_+ + \cot(\beta) h_-. \tag{20}$$

3.2 Study of The Properties of Hyperbolic Trigonometric Functions

Theorem 3.2.1: In the hyperbolic number plane $\mathbb{H}$, for any hyperbolic numbers $\delta_1, \delta_2 \in \mathbb{H}$, the hyperbolic sine function satisfies the addition formula:

$$\sin(\delta_1 + \delta_2) = \sin\delta_1 \cos\delta_2 + \cos\delta_1 \sin\delta_2. \tag{21}$$

Proof: Let $\forall \delta_1, \delta_2 \in \mathbb{H}$ .Within this framework, By the standard canonical representation, hyperbolic numbers are expressed as follows: $\delta_1 = \alpha_1 h_+ + \beta_1 h_- = x_0 + x_1 h,\ \delta_2 = \alpha_2 h_+ + \beta_2 h_- = y_0 + y_1 h.$ where the $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real coefficients, and h_+, h_- denote the basis elements of $\mathbb{H}$.

First, calculate the sum $\delta_1 + \delta_2$:

$$\delta_1 + \delta_2 = (\alpha_1 h_+ + \beta_1 h_-) + (\alpha_2 h_+ + \beta_2 h_-) = (\alpha_1 + \alpha_2) h_+ + (\beta_1 + \beta_2) h_-. \tag{22}$$

From which we obtain:

$$\begin{aligned} \sin(\delta_1 + \delta_2) &= \sin((\alpha_1 + \alpha_2) h_+ + (\beta_1 + \beta_2) h_-) \\ &= \sin((\alpha_1 + \alpha_2) h_+) + \sin((\beta_1 + \beta_2) h_-). \end{aligned} \tag{23}$$

Apply the addition formula for the sine function in real analysis to each component, it has:

$$\begin{aligned}\sin(\delta_1+\delta_2)&=\sin((\alpha_1+\alpha_2)h_+ +(\beta_1+\beta_2)h_-)\\&=(\sin(\alpha_1)\cos(\alpha_2)+\cos(\alpha_1)\sin(\alpha_2))h_+ +(\sin(\beta_1)\cos(\beta_2)+\cos(\beta_1)\sin(\beta_2))h_-.\end{aligned} \tag{24}$$

It can changed:

$$\sin(\delta_1+\delta_2)=(\sin(\alpha_1)\cos(\alpha_2)+\cos(\alpha_1)\sin(\alpha_2))h_+ + (\sin(\beta_1)\cos(\beta_2)+\cos(\beta_1)\sin(\beta_2))h_-, \tag{25}$$

$$\sin(\delta_1+\delta_2)=(\sin(\alpha_1)\cos(\alpha_2)h_+ +\sin(\beta_1)\cos(\beta_2)h_-)+ (\cos(\alpha_1)\sin(\alpha_2)h_+ +\cos(\beta_1)\sin(\beta_2)h_-), \tag{26}$$

$$\sin(\delta_1+\delta_2)=(\sin(\alpha_1)h_+ +\sin(\beta_1)h_-)(\cos(\alpha_2)h_+ +\cos(\beta_2)h_-)+ (\cos(\alpha_1)h_+ +\cos(\beta_1)h_-)(\sin(\alpha_2)h_+ +\sin(\beta_2)h_-). \tag{27}$$

Therefore, equation (21) is established.

Building upon the established angle addition formula for the hyperbolic sine function, now extend this analysis to the cosine function.

Theorem 3.2.2: In the hyperbolic number plane $\mathbb{H}$, for any hyperbolic numbers $\delta_1,\delta_2\in\mathbb{H}$, the hyperbolic cosine function satisfies the addition formula:

$$\cos(\delta_1+\delta_2)=\cos\delta_1\cos\delta_2-\sin\delta_1\sin\delta_2. \tag{28}$$

Proof: Let $\forall\delta_1,\delta_2\in\mathbb{H}$. By the canonical representation of hyperbolic numbers, this paper express them in as follows $\delta_1=\alpha_1h_+ +\beta_1h_- = x_0+x_1h,\ \delta_2=\alpha_2h_+ +\beta_2h_- = y_0+y_1h.$ where the are real coefficients, and denote the basis elements of $\mathbb{H}$.

From equation (22), it can be seen that:

$$\begin{aligned}\cos(\delta_1+\delta_2)&=\cos((\alpha_1+\alpha_2)h_+ +(\beta_1+\beta_2)h_-)\\&=\cos((\alpha_1+\alpha_2)h_+)+\cos((\beta_1+\beta_2)h_-).\end{aligned} \tag{29}$$

Apply the addition formula for the cosine function in real analysis to each component, it has:

$$\begin{aligned}\cos(\delta_1+\delta_2)&=\cos((\alpha_1+\alpha_2)h_+ +(\beta_1+\beta_2)h_-)\\&=(\cos(\alpha_1)\cos(\alpha_2)-\sin(\alpha_1)\sin(\alpha_2))h_+ +(\cos(\beta_1)\cos(\beta_2)-\sin(\beta_1)\sin(\beta_2))h_-.\end{aligned} \tag{30}$$

It can changed:

$$\cos(\delta_1+\delta_2)=(\cos(\alpha_1)\cos(\alpha_2)-\sin(\alpha_1)\sin(\alpha_2))h_+ + (\cos(\beta_1)\cos(\beta_2)-\sin(\beta_1)\sin(\beta_2))h_-, \tag{31}$$

$$\cos(\delta_1+\delta_2)=(\cos(\alpha_1)\cos(\alpha_2)h_+ +\cos(\beta_1)\cos(\beta_2)h_-)- (\sin(\alpha_1)\sin(\alpha_2)h_+ +\sin(\beta_1)\sin(\beta_2)h_-), \tag{32}$$

$$\cos(\delta_1+\delta_2)=(\cos(\alpha_1)h_+ +\cos(\beta_1)h_-)(\cos(\alpha_2)h_+ +\cos(\beta_2)h_-)- (\sin(\alpha_1)h_+ +\sin(\beta_1)h_-)(\sin(\alpha_2)h_+ +\sin(\beta_2)h_-). \tag{33}$$

Therefore, equation (28) is established.

Based on the results above, following the successful derivation of the angle addition formulas for both sine and cosine functions in the hyperbolic number plane, now complete this trigonometric foundation by examining the tangent function.

Theorem 3.2.3: Let $\delta_1,\delta_2\in\mathbb{H}$, $\delta_i=\alpha_ih_+ +\beta_ih_-\in\mathbb{H}$, and neither of values α and β equals $\frac{\pi}{2}+k\pi$. When $\cos\delta_1\cos\delta_2-\sin\delta_1\sin\delta_2$ is invertible , it has:

$$\tan(\delta_1+\delta_2)=\frac{\tan\delta_1+\tan\delta_2}{1-(\tan\delta_1\tan\delta_2)}. \tag{34}$$

But when $\cos\delta_1\cos\delta_2-\sin\delta_1\sin\delta_2$ is invertible , it has:

$$\tan(\delta_1+\delta_2)=\frac{\tan\alpha_1+\tan\alpha_2}{1-(\tan\alpha_1\tan\alpha_2)}h_+ +\frac{\tan\beta_1+\tan\beta_2}{1-(\tan\beta_1\tan\beta_2)}h_-. \tag{35}$$

Proof: $\forall\delta_1,\delta_2\in\mathbb{H}$, when $\cos\delta_1\cos\delta_2-\sin\delta_1\sin\delta_2$ is invertible. By the definition of the hyperbolic tangent function, $\tan\delta=\dfrac{\sin\delta}{\cos\delta}$. Thus, $\tan(\delta_1+\delta_2)=\dfrac{\sin(\delta_1+\delta_2)}{\cos(\delta_1+\delta_2)}$.

From Theorem 3.2.1 and Theorem 3.2.2, it follows that:

$$\tan(\delta_1+\delta_2)=\frac{\sin(\delta_1+\delta_2)}{\cos(\delta_1+\delta_2)}=\frac{\sin(\delta_1)\cos(\delta_2)+\cos(\delta_1)\sin(\delta_2)}{\cos(\delta_1)\cos(\delta_2)-\sin(\delta_1)\sin(\delta_2)}. \tag{36}$$

Divide both the numerator and the denominator by $\cos\delta_1\cos\delta_2$, it simplify to obtain (34).

When $\cos\delta_1\cos\delta_2-\sin\delta_1\sin\delta_2$ is not invertible. Let $\forall\delta_1=\alpha_1h_++\beta_1h_-$ and $\forall\delta_2=\alpha_2h_++\beta_2h_-$, express $\delta_1+\delta_2$ in terms of the idempotent basis :

$$\delta_1+\delta_2=(\alpha_1+\alpha_2)h_++(\beta_1+\beta_2)h_-. \tag{37}$$

From which we obtain:

$$\begin{aligned}\tan(\delta_1+\delta_2)&=\tan((\alpha_1+\alpha_2)h_++(\beta_1+\beta_2)h_-)\\&=\tan((\alpha_1+\alpha_2)h_+)+\tan((\beta_1+\beta_2)h_-).\end{aligned} \tag{38}$$

Apply the addition formula for the tangent function in real analysis to each component, it has:

$$\tan(\alpha_1+\alpha_2)=\frac{\sin(\alpha_1+\alpha_2)}{\cos(\alpha_1+\alpha_2)}=\frac{\tan\alpha_1+\tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2}, \tag{39}$$

$$\tan(\beta_1+\beta_2)=\frac{\sin(\beta_1+\beta_2)}{\cos(\beta_1+\beta_2)}=\frac{\tan\beta_1+\tan\beta_2}{1-\tan\beta_1\tan\beta_2}. \tag{40}$$

It can changed:

$$\begin{aligned}\tan(\delta_1+\delta_2)&=\tan((\alpha_1+\alpha_2)h_++(\beta_1+\beta_2)h_-)\\&=\tan(\alpha_1+\alpha_2)h_++\tan(\beta_1+\beta_2)h_-,\end{aligned} \tag{41}$$

$$\begin{aligned}\tan(\delta_1+\delta_2)&=\frac{\sin(\alpha_1+\alpha_2)}{\cos(\alpha_1+\alpha_2)}h_++\frac{\sin(\beta_1+\beta_2)}{\cos(\beta_1+\beta_2)}h_-\\&=\frac{\tan(\alpha_1)+\tan(\alpha_2)}{1-\tan(\alpha_1)\tan(\alpha_2)}h_++\frac{\tan(\beta_1)+\tan(\beta_2)}{1-\tan(\beta_1)\tan(\beta_2)}h_-.\end{aligned} \tag{42}$$

Therefore, equation (35) is established.

The hyperbolic cotangent is the reciprocal of the hyperbolic tangent. Building on the angle addition formula for tanh, now derive the corresponding formula for coth. This establishes the following theorem.

Theorem 3.2.4: Let $\delta_1,\delta_2\in\mathbb{H}$, $\delta_i=\alpha_ih_++\beta_ih_-\in\mathbb{H}$, and neither of values $\alpha=\sum\alpha_i$ and $\beta=\sum\beta_i$ equals $\dfrac{\pi}{2}+k\pi$. When $\sin\delta_1\cos\delta_2+\cos\delta_1\sin\delta_2$ is invertible , it has:

$$\cot(\delta_1+\delta_2)=\frac{\cot\delta_1\cot\delta_2-1}{\cot\delta_1+\cot\delta_2}. \tag{43}$$

But when $\sin\delta_1\cos\delta_2+\cos\delta_1\sin\delta_2$ is invertible , it has:

$$\cot(\delta_1+\delta_2)=\frac{\cot\alpha_1\cot\alpha_2-1}{\cot\alpha_1+\cot\alpha_2}h_++\frac{\cot\beta_1\cot\beta_2-1}{\cot\beta_1+\cot\beta_2}h_-. \tag{44}$$

Proof: $\forall\delta_1,\delta_2\in\mathbb{H}$, when $\sin\delta_1\cos\delta_2+\cos\delta_1\sin\delta_2$ is invertible. By the definition of the hyperbolic cotangent function, $\cot\delta=\dfrac{\cos\delta}{\sin\delta}$. Thus, $\cot(\delta_1+\delta_2)=\dfrac{\cos(\delta_1+\delta_2)}{\sin(\delta_1+\delta_2)}$.

From Theorem 3.2.1 and Theorem 3.2.2, it follows that:

$$\cot(\delta_1+\delta_2)=\frac{\cos(\delta_1+\delta_2)}{\sin(\delta_1+\delta_2)}=\frac{\cos(\delta_1)\cos(\delta_2)-\sin(\delta_1)\sin(\delta_2)}{\sin(\delta_1)\cos(\delta_2)+\cos(\delta_1)\sin(\delta_2)}. \tag{45}$$

Divide both the numerator and the denominator by $\sin\delta_1\sin\delta_2$, it simplify to obtain (43).

When $\sin\delta_1\cos\delta_2+\cos\delta_1\sin\delta_2$ is not invertible. Let $\forall\delta_1=\alpha_1h_++\beta_1h_-$ and $\forall\delta_2=\alpha_2h_++\beta_2h_-$, express $\delta_1+\delta_2$ in terms of the idempotent basis :

$$\delta_1+\delta_2=(\alpha_1+\alpha_2)h_++(\beta_1+\beta_2)h_-. \tag{46}$$

From which we obtain:

$$\begin{aligned}\cot(\delta_1+\delta_2)&=\cot((\alpha_1+\alpha_2)h_++(\beta_1+\beta_2)h_-)\\&=\cot((\alpha_1+\alpha_2)h_+)+\cot((\beta_1+\beta_2)h_-).\end{aligned} \tag{47}$$

Apply the addition formula for the cotangent function in real analysis to each component, it has:

$$\cot(\alpha_1+\alpha_2)=\frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)}=\frac{\cot\alpha_1\cot\alpha_2-1}{\cot\alpha_1+\cot\alpha_2}, \tag{48}$$

$$\cot(\beta_1+\beta_2)=\frac{\cos(\beta_1+\beta_2)}{\sin(\beta_1+\beta_2)}=\frac{\cot\beta_1\cot\beta_2-1}{\cot\beta_1+\cot\beta_2}. \tag{49}$$

It can changed:

$$\begin{aligned}\cot(\delta_1+\delta_2)&=\cot((\alpha_1+\alpha_2)h_++(\beta_1+\beta_2)h_-)\\&=\cot(\alpha_1+\alpha_2)h_++\cot(\beta_1+\beta_2)h_-,\end{aligned} \tag{50}$$

$$\begin{aligned}\cot(\delta_1+\delta_2)&=\frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)}h_++\frac{\cos(\beta_1+\beta_2)}{\sin(\beta_1+\beta_2)}h_-\\&=\frac{\cot\alpha_1\cot\alpha_2-1}{\cot\alpha_1+\cot\alpha_2}h_++\frac{\cot\beta_1\cot\beta_2-1}{\cot\beta_1+\cot\beta_2}h_-.\end{aligned} \tag{51}$$

Therefore, equation (44) is established.

4 CONCLUSIONS

This paper has conducted a systematic investigation into the fundamental properties of hyperbolic trigonometric functions defined on the hyperbolic number plane. The core achievement lies in the rigorous establishment of the addition theorems for the four primary hyperbolic trigonometric functions: sine, cosine, tangent, and cotangent, as detailed in Theorems 3.2.1 through Theorems 3.2.4. The successful derivation of these formulas relied critically on leveraging the unique decomposition properties afforded by the idempotent basis of hyperbolic numbers, which effectively circumvented the analytical challenges posed by the presence of zero divisors in the underlying algebra. The proven addition theorems constitute a fundamental and complete set of identities for hyperbolic trigonometric functions, significantly enriching their theoretical framework. These results not only solidify the theoretical foundation of hyperbolic function theory but also provide indispensable analytical tools. They are poised to facilitate and stimulate further research in hyperbolic analysis, particularly in the study of hyperbolic differential equations, integral transforms, and the convergence properties of hyperbolic trigonometric series. Ultimately, this work opens several promising avenues for future research. The established properties are expected to serve as a cornerstone for more advanced analytical developments within the hyperbolic number system and to find meaningful applications in related physical and geometric contexts where the structure of hyperbolic numbers proves natural and powerful.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

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