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AUTOMATIC GENERATION OF AESTHETIC CHAOTIC ATTRACTORS WITH MULTIPLE CYCLIC AND DIHEDRAL SYMMETRIES

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Abstract: This paper presents a novel and robust computational framework for the automatic generation of visually compelling chaotic attractors that exhibit multiple, distinct cyclic or dihedral symmetries within a single image plane. Building upon the foundational concept of equivariant functions, our approach ingeniously combines normalization, geometric translation, and scale transformation techniques to partition the phase space into concentric ring-like regions, each governed by its own symmetry group. To address the critical challenge of rendering aesthetically pleasing visualizations from the inherently complex and dense orbit data, we introduce a significant enhancement to the traditional frequency-based coloring scheme. Our improved algorithm employs a color percentage distribution technique, which allows for precise, priori control over the color palette and its spatial allocation in the final image. This not only streamlines the artistic creation process but also provides a more consistent and controllable method for scientific visualization. The proposed methodology is computationally efficient and highly versatile, capable of producing a vast array of intricate and beautiful patterns. We provide detailed implementation protocols, including specific parameter sets and pseudocode, to ensure full reproducibility. The results demonstrate that our system can reliably generate high-resolution images showcasing up to four-fold or higher symmetries, constrained primarily by computational resolution limits, thereby opening new avenues for both mathematical art and the study of symmetric dynamical systems.
Keywords: Chaotic attractors; Symmetry groups; Cyclic symmetry; Dihedral symmetry; Equivariant functions; Orbit rendering; Computational aesthetics

1 INTRODUCTION

The intersection of chaos theory and symmetry has long fascinated mathematicians, physicists, and artists alike. Chaotic systems, characterized by their sensitive dependence on initial conditions and seemingly random long-term behavior, often give rise to strange attractors—geometric structures that are both infinitely complex and possess an underlying order. Remarkably, this order frequently manifests as symmetry, a fundamental concept in mathematics and nature [1]. Discrete symmetry groups in the Euclidean plane are well-classified, with cyclic and dihedral groups representing the simplest yet most pervasive forms, governing the rotational and reflectional symmetries of countless natural and man-made objects [2].

Recent decades have seen a surge of interest in deliberately constructing dynamical systems whose attractors embody specific symmetries. Pioneering work by Field and Golubitsky established a rigorous mathematical foundation for generating chaotic attractors with prescribed symmetries using equivariant mappings [3]. Subsequent research has expanded this domain to include more complex wallpaper and frieze groups and has developed sophisticated rendering techniques like the orbit trap method to translate the abstract dynamics into vibrant, colored artworks, see Figure 1 [4-7].

Figure 1 Trolley Hub with Rotational Symmetry

However, a significant gap remains in the literature regarding the generation of attractors that seamlessly integrate *multiple* distinct symmetries within a unified visual field. Most existing methods produce attractors with a single, global symmetry. In this work, we bridge this gap by proposing a novel algorithmic pipeline. Our core innovation lies in the strategic use of geometric transformations—not to alter the fundamental dynamics, but to strategically map different regions of the attractor's basin of attraction onto distinct annular zones, each of which is then processed to exhibit its designated symmetry (e.g., C3 in the inner ring, D4 in the middle, and C5 in the outer ring). This creates a composite image where symmetry itself becomes a dynamic, spatially varying property.

Furthermore, we recognize that the aesthetic quality of the final image is as crucial as its mathematical correctness. The standard practice of coloring pixels based on visitation frequency often yields muddy or unbalanced visuals, requiring extensive manual parameter tuning. To overcome this, we develop and implement an advanced rendering algorithm based on color percentage distribution. This method decouples the raw data density from the final color assignment, allowing the artist or scientist to define a target histogram for the color channels, resulting in consistently striking and balanced compositions. The following sections detail our theoretical framework, the step-by-step generation and rendering algorithms, and present a gallery of results that validate the power and flexibility of our approach.

2 THEORETICAL FOUNDATIONS: SYMMETRY GROUPS AND EQUIVARIANT FUNCTIONS

Our methodology rests on two key mathematical pillars: the algebraic structure of symmetry groups and the dynamical property of equivariance. A thorough understanding of these concepts is essential for the deliberate design of chaotic systems with prescribed geometric properties.

2.1 Cyclic and Dihedral Groups

In the context of planar geometry, a **cyclic group** of order n, denoted C_n, consists of all rotations about a fixed point by angles that are integer multiples of $2\pi/n$. It is generated by a single rotation operation. A **dihedral group** of order $2n$, denoted D_n, extends C_n by including n reflection symmetries across lines passing through the same central point. Thus, $D_n = C_n \cup \{\text{reflections}\}$, capturing the full symmetry of a regular n-gon. These groups form the foundation for describing the rotational and reflectional invariance observed in our target attractors.

2.2 Constructing Symmetric Attractors via Equivariance

To ensure that the attractor of a discrete dynamical system $z_{k+1} = f(z_k)$ possesses the symmetry of a group Γ, the iteration function f must be **Γ-equivariant**. This means that for every symmetry operation $\gamma \in \Gamma$, the function commutes with γ: $f(\gamma z) = \gamma f(z)$ for all points z in the plane. This condition guarantees that if an orbit passes through a point z, it must also pass through all its symmetric counterparts γz, thereby forcing the entire attractor to be invariant under the action of Γ.

Following the seminal approach of Field and Golubitsky [3] and its practical implementation by Jones and Reiter [8], we construct our base chaotic function in the complex plane as a sum of rotated and scaled versions of an arbitrary, typically non-symmetric, "seed" function $g(z)$. For a desired cyclic symmetry C_n, the equivariant function is:

$$f_{C_n}(z) = \Sigma_{j=0}^{n-1}\, \omega^{-j}\, g(\omega^j z) \quad (1)$$

where $\omega = e^{2\pi i/n}$ is a primitive n-th root of unity. This construction is not merely a mathematical curiosity; it is a powerful design principle. By choosing a seed function g that is known to produce chaotic behavior (e.g., rational functions like $g(z) = (az + b)/(cz + d)$ or transcendental functions like $g(z) = \sin(z) + c$), the averaging process inherent in the summation preserves the chaotic nature while imposing the desired global symmetry.

Advantages of this Method: This approach offers several significant advantages. First, it provides a systematic and general recipe for generating attractors with any prescribed cyclic or dihedral symmetry, which is far more flexible than attempting to reverse-engineer a polynomial with specific symmetry properties. Second, the choice of the seed function g allows for immense creative control over the fine-grained structure and texture of the attractor, enabling the exploration of a vast design space. Third, the method is computationally straightforward to implement, requiring only basic complex arithmetic and loops.

Limitations and Challenges: However, this method is not without its drawbacks. The primary challenge is **parameter sensitivity**. The chaotic behavior and visual appeal of the resulting attractor are highly dependent on the parameters within the seed function g and the constant multipliers in the sum. Finding aesthetically pleasing parameter sets often requires extensive numerical experimentation or sophisticated search algorithms [9]. A second limitation is the potential for **numerical instability**. For certain seed functions and parameter choices, the iterated map can diverge to infinity for most initial conditions, yielding an empty or trivial attractor. Careful selection of the seed function and the use of bailout conditions during iteration are necessary to mitigate this. Finally, while the method excels at creating attractors with a single, global symmetry, it is fundamentally incompatible with generating a single, connected attractor that exhibits multiple, distinct global symmetries (e.g., both C3 and C4), as their defining group actions are mutually exclusive. This inherent limitation is precisely the problem our spatially segmented approach in Section 3 aims to circumvent.

Recent research has sought to address some of these limitations. Deep learning techniques have been explored to predict chaotic parameters from desired visual features [10], and hybrid models combining equivariant maps with neural networks have shown promise in stabilizing dynamics [11]. Furthermore, the study of symmetry-breaking bifurcations in such systems provides insights into how complex, asymmetric patterns can emerge from symmetric initial conditions

[12], adding another layer of complexity to the design process. Our work builds upon this evolving landscape by offering a novel architectural solution to the multi-symmetry problem.

3 ALGORITHM FOR GENERATING ATTRACTORS WITH MULTIPLE SYMMETRIES

A single image cannot simultaneously possess, for example, both C3 and C4 symmetry globally, as their rotational requirements are incompatible. Our solution is to create a spatially segmented attractor. The algorithm Φ1 proceeds as follows:

1. Define Symmetry Zones: Partition the image canvas into m concentric annular regions, R_1, ..., R_m, each assigned a specific symmetry group Γ_i (e.g., C3, D5, C7).
2. Generate Base Orbits: Select a seed function $g(z)$ and iterate the dynamical system $z_{k+1} = g(z_k)$ for a large number of steps (e.g., 10^7), discarding an initial transient period. This produces a long orbit $\{z_k\}$ that densely populates the base attractor.
3. Normalize and Map Orbits: For each point z_k in the orbit:

a. Normalize: Compute its magnitude $r = |z_k|$ and normalize it to a unit interval, $r_{norm} = r / r_{max}$, where r_{max} is a chosen cutoff radius.
b. Assign Zone: Determine which annular region R_i the point belongs to based on r_{norm}.
c. Apply Symmetry Transformation: Transform the point z_k into the canonical coordinate system of its assigned zone. This involves translating the origin to the center of the annulus and scaling the radial coordinate so that the annulus maps to a standard unit ring. Let this transformed point be w_k.
d. Render with Local Symmetry: Apply the equivariant function $f_{\Gamma i}$ corresponding to the zone's symmetry to w_k. The resulting point is then mapped back to the original image coordinates and recorded for the final rendering step.

This process effectively "paints" different parts of the base chaotic orbit with different symmetry filters, creating a composite image where the type of symmetry changes as one moves radially outward from the center, see Figure 2.

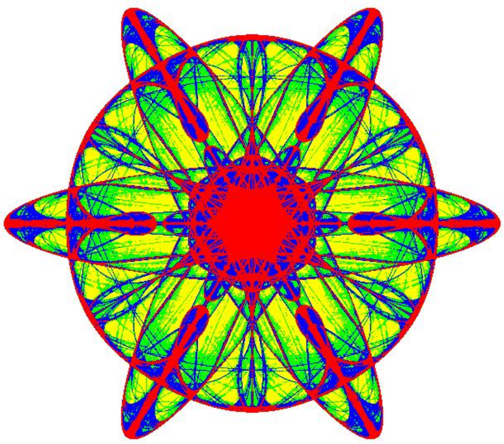

Figure 2 Chaotic Attractor with D6 Symmetry

3.1 Comparative Analysis with Alternative Approaches

It is instructive to contrast the equivariant summation method with two other prominent strategies for generating symmetric chaotic attractors: polynomial-based construction and symmetry-enforced post-processing.

The polynomial-based approach, historically used in early studies of symmetric chaos [13], seeks to directly define a map as a complex polynomial whose algebraic form inherently respects a given symmetry group. While mathematically elegant and analytically tractable for low-order symmetries, this method suffers from severe limitations in practice. High-degree polynomials often lead to rapidly diverging orbits or overly regular, non-chaotic structures. Moreover, designing a polynomial that simultaneously exhibits rich chaotic dynamics and a specific high-order symmetry (e.g., D_7) is non-trivial and lacks a general constructive procedure. In contrast, the equivariant summation method decouples the chaos-generating mechanism (the seed function $f_{Cn}(z) = \Sigma_{j=0}^{n-1} \omega^{-j} g(\omega^j z)$) from the symmetry-imposing mechanism (the group averaging), offering far greater flexibility and robustness.

The second alternative—symmetry-enforced post-processing—involves first generating a standard chaotic attractor (without any symmetry) and then applying geometric symmetrization (e.g., rotating and superimposing copies) during the rendering phase. While simple to implement, this approach is fundamentally flawed from a dynamical systems perspective. The resulting image may look symmetric, but it does not correspond to the true invariant set of any underlying Γ-equivariant dynamical system. The orbits are not genuinely constrained by the symmetry; instead, symmetry is merely an optical illusion imposed after the fact. This breaks the deep connection between the geometry of the attractor and the algebraic structure of the governing equations—a connection that is central to the theory of symmetric chaos [3,12]. Our equivariant method, by contrast, ensures that symmetry is an intrinsic property of the dynamics itself, not an artifact of visualization.

Therefore, while polynomial methods are restrictive and post-processing methods are dynamically inconsistent, the equivariant summation framework strikes an optimal balance: it is general enough to accommodate a wide range of chaotic seeds, mathematically rigorous in its enforcement of symmetry, and computationally efficient. This makes it the most suitable foundation upon which to build our proposed multi-symmetry architecture, see Figure 3.

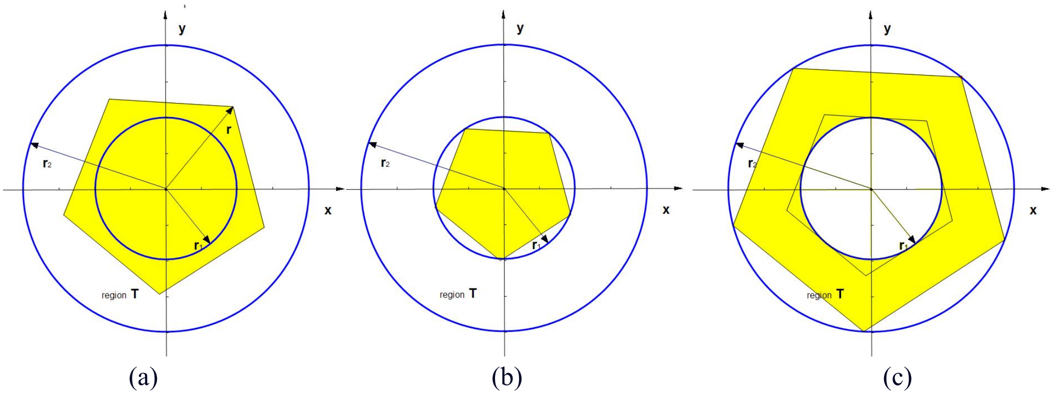

Figure 3 Schematic diagram of algorithm Φ1

4 IMPROVED RENDERING ALGORITHM VIA COLOR PERCENTAGE DISTRIBUTION

The traditional method of coloring a pixel based on how many times an orbit visits it often leads to a few very bright "hot spots" and large dark areas, which is visually unappealing. Our improved algorithm aims for a more balanced and controllable aesthetic.

1. Accumulate Visit Counts: As in the traditional method, we first create a 2D histogram (a grid of counters) where each cell corresponds to a pixel and counts the number of orbit visits.
2. Compute Global Statistics: After the orbit is complete, we analyze the entire histogram to find the minimum ($C_{\min}$) and maximum ($C_{\max}$) visit counts.
3. Define Target Color Distribution: Instead of a linear map from [$C_{\min}$, $C_{\max}$] to a color gradient, we define a target cumulative distribution function (CDF) for the colors. For instance, we might want the darkest 10% of pixels to be black, the next 30% to transition through blue, the next 40% through green to yellow, and the brightest 20% to be white.
4. Map Counts to Colors: We sort all non-zero pixel counts and assign colors according to our pre-defined target CDF. This ensures that the final image uses the full range of the chosen color palette in a balanced way, regardless of the underlying, potentially skewed, visitation statistics of the chaotic orbit. This technique provides the user with direct, intuitive control over the final look of the artwork, see Figure 4.

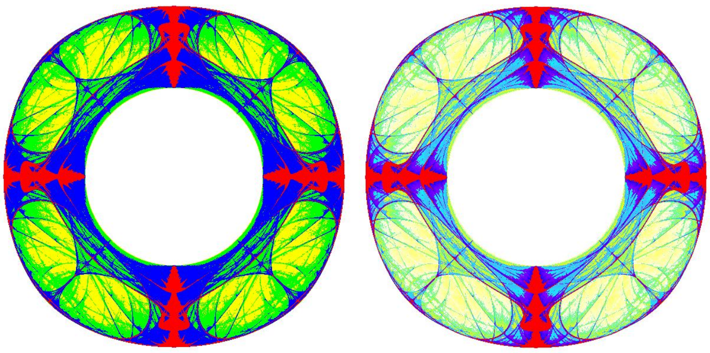

Figure 4 Chaotic Attractor with D4 Symmetry Generated by Different Palette

5 EXAMPLES AND IMPLEMENTATION

We implemented our algorithm in Python using NumPy for computation and Matplotlib/PIL for rendering. A typical run uses a seed function like $g(z) = z + \sin(z)/z$ and iterates for 5 million points after a 10,000-point transient. The leftmost of Figure 5 displays an attractor with two zones: an inner disk with C3 symmetry and an outer ring with C8 symmetry, rendered using our color distribution method with a "plasma" colormap. The result is a harmonious blend of order and chaos, with clear, distinct symmetry patterns in each region. The primary limitation is the finite pixel resolution of digital displays; theoretically, the method can accommodate any number of symmetries, but in practice, the finest details of high-order symmetries (e.g., C20) may become indistinguishable at standard resolutions, see the rightmost of Figure 5.

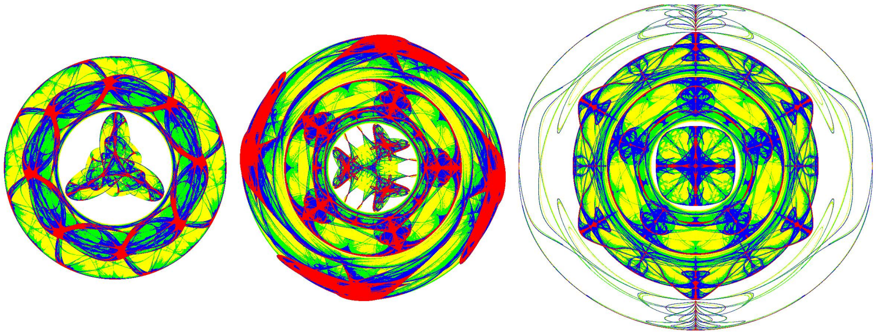

Figure 5 From Left to Right, C3_C6, C4_D5_D3 and D2_D6_D5_4 Attractors Obtained by Combining Several Basic Attractors

6 CONCLUSION

In this paper, we have successfully developed and demonstrated a comprehensive framework for the automatic generation of chaotic attractors that exhibit multiple, spatially segregated cyclic and dihedral symmetries. By combining the mathematical rigor of equivariant functions with a clever geometric partitioning strategy, we have overcome the incompatibility of global multiple symmetries. Furthermore, our novel color percentage distribution rendering algorithm provides unprecedented control over the aesthetic output, transforming chaotic data into predictable and beautiful visual compositions. This work not only contributes to the field of mathematical art but also offers a powerful new tool for visualizing and understanding the complex interplay between symmetry and chaos in dynamical systems. Future work will explore the extension of this method to three-dimensional attractors and other, more complex symmetry groups.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

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