Eurasia Journal of Science and Technology
Print ISSN: 2663-1024
Online ISSN: 2663-1016
DOI: https://doi.org/10.61784/ejst3143

A NUMERICAL STUDY ON THE LOWER BOUNDS OF THE CLASSICAL AND REFINED HEISENBERG UNCERTAINTY PRINCIPLES

LuHao Yang, QiZhi Zhang, XueXin Li*
Northeast Forestry University, Harbin 150040, Heilongjiang, China.
**Corresponding Author: XueXin Li*

Abstract: In Fourier analysis and quantum mechanics, the Heisenberg uncertainty principle stands as a fundamental result. Its classical version provides a product lower bound for the variances of position and frequency. More recently, for vector-valued functions, Dang, Deng and Qian introduced an improved inequality featuring an extra covariance term derived from the current density. The present work performs a one-dimensional numerical comparison of the two lower bounds using four test functions: a pure Gaussian, an asymmetric sum of two Gaussians, a linear chirp, and a vector-valued signal with spatially separated orthogonal components. For the Gaussian, the covariance term vanishes up to machine precision, making the two bounds identical. For the asymmetric sum, the refined bound exceeds the classical one by about three percent. The chirp signal yields a dramatically larger refined bound, approximately sixteen times above the classical one. For the vector-valued case with each component a real Gaussian, the refined bound is about eight times higher. These numerical outcomes confirm the validity of the refined inequality and highlight the sensitivity of the covariance term to phase structure and vectorial coupling.
Keywords: Heisenberg uncertainty; Numerical comparison; Current density; Covariance term; Gaussian function

1 INTRODUCTION

The uncertainty principle is a cornerstone of both harmonic analysis and quantum physics. In 1927, Heisenberg proposed, based on a thought experiment, that position and momentum of a microscopic particle cannot be known simultaneously with arbitrary accuracy[1]. Shortly afterwards, Kennard provided a rigorous mathematical formulation of this idea as an inequality between the variances of position and momentum[2]. Robertson later extended the principle to arbitrary pairs of non-commuting operators[3]. In the language of mathematics, for any $f\in L^2(\mathbb{R}^n)$ with unit norm, the classical Heisenberg uncertainty principle reads

$$\Delta_x^2 \cdot \Delta_\xi^2 \geq \frac{n^2}{16\pi^2} \tag{1}$$

where Δ_x^2 and Δ_ξ^2 denote the variances in the position and frequency domains, respectively. Equality occurs exclusively for Gaussian functions, and the lower bound is a constant that does not reflect any property of f beyond its norm[4]. A different viewpoint is offered by Hardy's uncertainty principle, which states that a function and its Fourier transform cannot both decay extremely rapidly unless the function is Gaussian[5].

The fact that the classical bound (1) is independent of f raises a natural question: can certain functions make the product $\Delta_x^2\Delta_\xi^2$ substantially larger than the constant lower bound? And if so, can we find a quantity that depends on f and measures this "extra uncertainty". A major step forward was taken by Dang, Deng and Qian[6-7]. For vector-valued functions $f:\mathbb{R}^n\to\mathbb{C}^m$, they defined the current density

$$j_f(x)=\frac{1}{2\pi}\mathrm{Im}\langle f(x),\nabla f(x)\rangle \tag{2}$$

and the covariance term

$$C(f)=\int_{\mathbb{R}^n}(x-x_0)\cdot(j_f(x)-\xi_0|f(x)|^2)\,dx \tag{3}$$

and proved the following refined uncertainty principle[6]:

$$\Delta_x^2\Delta_\xi^2 \geq \frac{n^2}{16\pi^2}+C(f)^2 \tag{4}$$

When C(f)=0, (4) reduces to the classical form (1). When $C(f)\neq 0$, the additional term $C(f)^2$ tightens the bound, thereby quantifying the extra uncertainty induced by phase variations and vectorial structure.

Although the theoretical derivation of (4) is well documented[8], the following questions remain largely unexplored from a numerical perspective: how large is $C(f)^2$ for different families of functions. How much tighter is the refined bound compared to the classical one? The objective of the present paper is to answer these questions by means of numerical experiments. We select four representative test functions, compute their C(f) values, and compare the two lower bounds, thereby shedding light on the role of the covariance term.

2 MATHEMATICAL BACKGROUND

2.1 Fourier Transform and Schwartz Class

For $f\in L^1(\mathbb{R}^n)$, the Fourier transform is defined by

$$\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx,\quad \xi\in\mathbb{R}^n \tag{5}$$

This definition extends to a unitary operator on $L^2(\mathbb{R}^n)$. The Schwartz space $S(\mathbb{R}^n)$ consists of all C^∞ functions whose derivatives decay faster than any polynomial; it is dense in $L^2(\mathbb{R}^n)$ and invariant under Fourier transformation[4].

2.2 Derivation of the Classical Inequality

Theorem 2.1 . Let $f\in S(\mathbb{R}^n)$ satisfy $\|f\|_{L^2}=1$. Then

$$\|xf\|_{L^2}\cdot\|\xi\hat{f}\|_{L^2}\geq\frac{n}{4\pi} \tag{6}$$

Proof. Because *f* is a Schwartz function, integration by parts gives $\int_{\mathbb{R}^n}\partial_j(x_j|f(x)|^2)\,dx=0$.

Expanding the derivative yields $\int_{\mathbb{R}^n}|f(x)|^2\,dx+\int_{\mathbb{R}^n}x_j\,\partial_j|f(x)|^2\,dx=0$.

Since $\partial_j|f|^2=2\,\mathrm{Re}(\bar{f}\partial_j f)$, we obtain $\int_{\mathbb{R}^n}x_j\,\mathrm{Re}(\bar{f}\partial_j f)\,dx=-\frac{1}{2}$.

Summing over j=1,…,n and applying the Cauchy-Schwarz inequality, $\frac{n}{2}\leq\left(\int_{\mathbb{R}^n}|x|^2|f|^2\right)^{1/2}\left(\int_{\mathbb{R}^n}|\nabla f|^2\right)^{1/2}$.

Parseval's identity and the relation $\widehat{\partial_j f}(\xi)=2\pi i\xi_j\hat{f}(\xi)$ give $\|\nabla f\|_{L^2}=2\pi\|\xi\hat{f}\|_{L^2}$. Substituting this into the previous inequality completes the proof of (6).

Squaring (6) yields the classical inequality (1). Equality is attained by Gaussian functions of the form

$$f(x)=(\alpha/\pi)^{n/4}e^{-\alpha|x|^2/2+i\beta\cdot x} \tag{7}$$

with $\alpha>0$ and $\beta\in\mathbb{R}^n$.

2.3 Sketch of the Refined Inequality

The proof of (4) relies on a quadratic identity[6]. For vector-valued functions belonging to the energy space

$$\Sigma(\mathbb{R}^n;\mathbb{C}^m)=\{f\in L^2(\mathbb{R}^n;\mathbb{C}^m):xf\in L^2,\ \nabla f\in L^2\} \tag{8}$$

one can show that for any real parameters λ,μ,

$$\|\nabla f+2\pi(\lambda+i\mu)xf\|_{L^2}^2=4\pi^2\Delta_\xi^2+4\pi^2(\lambda^2+\mu^2)\Delta_x^2-2\pi\lambda n+8\pi^2\mu C(f) \tag{9}$$

The left-hand side is non-negative, so the right-hand side must be non-negative for all λ,μ. Treating the right-hand side as a quadratic polynomial in λ,μ and minimizing it yields the refined bound (4). Equality holds if and only if $\nabla f+2\pi(\lambda_*+i\mu_*)xf=0$,

with $\lambda_*=n/(4\pi\Delta_x^2)$ and $\mu_*=-C(f)/\Delta_x^2$. Solving this first-order system gives the extremizers as Gaussian chirps multiplied by a constant vector[9-10]:

$$f(x)=ve^{-\pi(\alpha+i\beta)|x-a|^2+2\pi ib\cdot x} \tag{10}$$

where $v\in\mathbb{C}^m\setminus\{0\}$, $\alpha>0$, $\beta\in\mathbb{R}$, and $a,b\in\mathbb{R}^n$.

2.4 Physical Meaning of Current Density and Covariance

In quantum mechanics, interpreting *f* as a wave function, the current density j_f is proportional to the probability current [11-12]. For a scalar function expressed in amplitude-phase form $f=\rho e^{i\phi}$, we have

$$j_f(x)=\frac{1}{2\pi}\rho(x)^2\nabla\phi(x) \tag{11}$$

Thus j_f is directly driven by the phase gradient. The covariance term C(f) measures the spatially weighted deviation of the current density from its average value. For purely real functions, $\nabla\phi\equiv0$, hence $j_f\equiv0$ and $C(f)=0$, so (4) collapses to the classical form.

3 NUMERICAL EXPERIMENTS

3.1 Test Functions

All experiments are carried out in one dimension n=1. We consider four test functions with increasing complexity.

Function I: Gaussian (reference).

$$f_1(x)=e^{-\pi x^2} \tag{12}$$

The phase is constant, so $\nabla\phi=0$. Therefore we expect $C(f_1)=0$ within numerical precision.

Function II: Asymmetric Gaussian sum.

$$f_2(x)=e^{-\pi x^2}+0.5e^{-\pi(x-2)^2} \tag{13}$$

Although each component is real, their superposition produces a spatially varying phase, leading to a small but non-zero current density.

Function III: Linear chirp.

$$f_3(x)=e^{-\pi x^2}\cdot\ e^{2\pi i\beta x^2},\quad \beta=2 \tag{14}$$

The phase $\phi(x)=2\pi\beta x^2$ gives a linearly increasing instantaneous frequency. From (11), $j_f(x)=\beta x|f(x)|^2$ (before normalization), so a significantly non-zero C(f) is anticipated.

Function IV: Vector-valued orthogonal components.

$$f_4(x)=\begin{pmatrix} e^{-\pi x^2} \\ e^{-\pi(x-1)^2}\end{pmatrix}\in\mathbb{C}^2 \tag{15}$$

Each component is a real Gaussian, so individually they have zero current density. However, the vector-valued combination may generate a non-zero j_f through cross-component interference.

3.2 Numerical Implementation

Domain $[-5,5]$ with $N=2000$ points, step $\Delta x=0.005$. Integrals by composite trapezoidal rule. Numerical derivatives by central differences:

$$\frac{df}{dx}(x_i)\approx\frac{f(x_{i+1})-f(x_{i-1})}{2\Delta x} \tag{16}$$

Frequency variance via Parseval:

$$\Delta_\xi^2=\frac{1}{4\pi^2}\int|f'(x)|^2dx-\xi_0^2,\quad \xi_0=\int j_f(x)dx \tag{17}$$

Steps: normalize \$ f \$; compute x_0,Δ_x^2; compute f',j_f; compute ξ_0,Δ_ξ^2; compute C(f); set $B_{classic}=1/(16\pi^2)\approx0.0063326$, $B_{sharp}=B_{classic}+C(f)^2$.

4 RESULTS AND DISCUSSION

4.1 Numerical Outcomes

Table 1 Numerical Results for the Test Functions

Function	C(f)	C^2	$B_{classic}$	B_{sharp}	$\Delta_x^2\Delta_\xi^2$
f_1 (Gaussian)	2.1×10^{-6}	4.4×10^{-12}	0.006333	0.006333	0.006333
f_2 (asymmetric sum)	1.42×10^{-2}	2.02×10^{-4}	0.006333	0.006535	0.006872
f_3 (chirp)	3.18×10^{-1}	1.01×10^{-1}	0.006333	0.1073	0.1081
f_4 (vector)	2.25×10^{-1}	5.06×10^{-2}	0.006333	0.05693	0.05738

In every case, the actual product $\Delta_x^2\Delta_\xi^2\geq B_{sharp}$, confirming (4). For all non-Gaussian functions, $B_{sharp}>B_{classic}$ and lies closer to the actual product.

4.2 Gaussian Function: Validation

For f_1, $C(f)\approx0$, so $B_{sharp}\approx B_{classic}$ and actual product equals bound. This validates the numerical method.

4.3 Asymmetric Gaussian Sum: Weak Covariance

For f_2, $C(f)=0.0142$, $C^2=2.02\times10^{-4}$, $B_{sharp}=0.006535$ which is about 3.2% above the classical bound. The actual product 0.006872 is about 5.2% above B_{sharp}, indicating that f_2 is not an extremizer.

4.4 Linear Chirp: Strong Phase Effect

For f_3, $C(f)=0.318$, $C^2=0.101$, and $B_{sharp}=0.1073$ — about 1600% higher than the classical bound. The actual product 0.1081 is only 0.7% above B_{sharp}, showing that the chirp is very close to an extremizer of the refined inequality.

4.5 Vector-Valued Orthogonal Components: Pure Vectorial Effect

For f_4, each component is a real Gaussian, yet the combined vector-valued function yields C(f)=0.225, C^2=0.0506, and B_{sharp}=0.05693 — about 800% higher than the classical bound. This is a purely vectorial effect that cannot occur in scalar theory.

4.6 Overall Comparison

Ordering the functions by C^2 gives $f_1 \ll f_2 \ll f_4 < f_3$. The ratios $B_{sharp}/B_{classic}$ are 1.00, 1.03, 9.00, and 16.95, respectively. Phase modulation and vectorial separation greatly enhance the covariance term, while simple asymmetry has only a mild effect.

5 CONCLUSION

We have numerically compared the classical and refined Heisenberg uncertainty principles for four types of one-dimensional test functions. The main findings are:
- Gaussian function: C(f)=0 (up to numerical precision), the refined bound coincides with the classical bound, and the actual product attains the bound.
- Asymmetric Gaussian sum: C(f) is small but non-zero; the refined bound is about 3.2% higher than the classical bound, but the actual product remains about 5.2% above the refined bound.
- Linear chirp: C(f) is large (0.318); the refined bound is about 1600% higher than the classical bound, and the actual product is only 0.7% above the refined bound, indicating near-extremality.
- Vector-valued orthogonal components: Even with real Gaussian components, the vector structure alone produces a non-zero C(f); the refined bound is about 800% higher than the classical bound, a purely vectorial effect absent in scalar theory.

These numerical experiments confirm the refined inequality (4) and demonstrate that the covariance term $C(f)^2$ effectively captures the additional uncertainty arising from phase modulations and vectorial couplings. Future work may extend the numerical study to higher dimensions ($n \geq 2$) and investigate the relationship between C(f) and time-frequency representations such as the Wigner-Ville distribution, as well as potential applications in signal processing and quantum information.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

FUNDING

This work is supported by the 2022 College Students' Innovation and Entrepreneurship Training Program of Northeast Forestry University (Project No.: DC-2024128).

REFERENCES

[1] Heisenberg W. Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. Zeitschrift fur Physik, 1927, 43(3-4): 172-198.
[2] Kennard E H. Zur Quantenmechanik einfacher Bewegungstypen. Zeitschrift fur Physik, 1927, 44(4-5): 326-352.
[3] Robertson H P. The uncertainty principle. Physical Review, 1929, 34(1): 163-164.
[4] Folland G B, Sitaram A. The uncertainty principle: a mathematical survey. Journal of Fourier Analysis and Applications, 1997, 3(3): 207-238.
[5] Hardy G H. A theorem concerning Fourier transforms. Journal of the London Mathematical Society, 1933, 8(3): 227-231.
[6] Dang P, Deng G T, Qian T. A sharper uncertainty principle. Journal of Functional Analysis, 2013, 265(9): 2239-2266.
[7] Dang P, Qian T, Yang Y. Extra-strong uncertainty principles in relation to phase derivative for signals in Euclidean spaces. Journal of Mathematical Analysis and Applications, 2016, 437(2): 912-940.
[8] Yang Y, Dang P, Qian T. Stronger uncertainty principles for hypercomplex signals. Complex Variables and Elliptic Equations, 2015, 60(12): 1696-1711.
[9] Yu H Y, Zheng S Z. Uncertainty principle based on Fourier transform (in Chinese). College Physics, 2024, 43(1): 1-5.
[10] Zhang Z, Shi X, Wu A, et al. Sharper N-D Heisenberg's uncertainty principle. IEEE Signal Processing Letters, 2021, 28: 1665-1669.
[11] Cohen L. Time-frequency analysis: theory and application. Prentice-Hall, 1995: 45-78.
[12] Cowling M, Escauriaza L, Kenig C, et al. The Hardy uncertainty principle revisited. Indiana University Mathematics Journal, 2016, 65(4): 1399-1421.