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A CONVERGENCE THEOREM FOR THE JACOBI AND GAUSS-SEIDEL ITERATIONS

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Abstract: This paper treats linear systems whose coefficient matrices are neither strictly diagonally dominant nor symmetric positive definite. By applying a matrix transformation, such a system can be converted into one with a symmetric positive definite coefficient matrix. Sufficient conditions are then established for the convergence of the Jacobi and Gauss–Seidel iterations. Numerical experiments are provided to demonstrate the validity of the theorem.
Keywords: Strictly diagonally dominant; Symmetric positive definite; Iterative method; Convergence; Numerical experiment

1 INTRODUCTION AND PRELIMINARIES

The principal iterative methods for solving linear systems include the Jacobi, Gauss–Seidel and SOR methods, and their convergence has long attracted considerable attention. When the coefficient matrix is strictly diagonally dominant, it is well known that both the Jacobi and the Gauss–Seidel methods converge [1]. Because the condition of strict diagonal dominance is rather restrictive, efforts have been made to weaken it. A new criterion was proposed in Lin's research and later improved in Liu's research [2-4]. Meanwhile, the convergence of iterative methods under various conditions on the coefficient matrix has also been proved [5-7]. For non-diagonally dominant tridiagonal systems, Liu Yizhong transformed the matrix into a pentadiagonal system and established the symmetric positive definiteness of the resulting coefficient matrix [8]. It is already known that the Gauss–Seidel iteration converges when the coefficient matrix is symmetric positive definite [9]. In this paper we shall give a new convergence theorem for the Jacobi and Gauss–Seidel methods.

Consider the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n, \end{cases} \tag{1}$$

whose coefficient matrix is nonsingular, so that the system has a unique solution. Denote

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then system (1) can be written compactly as: $AX = B$.

Decompose the coefficient matrix A into its diagonal part D, strictly lower triangular part L and strictly upper triangular part U, so that $A = D + L + U$, the system $AX = B$ becomes $(D + L + U)X = B$. If only the diagonal part is retained on the left-hand side and the remaining terms are moved to the right, we obtain the pseudo-diagonal form $DX = -(L+U)X + B$, which yields $X = -D^{-1}(L+U)X + D^{-1}B.$

Based on this, the iterative method

$$X^{(k+1)} = -D^{-1}(L+U)X^k + D^{-1}B,$$

is constructed, with the component-wise formula

$$\begin{cases} X^{(0)} = & \left(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}\right)^T \\ x_i^{(k+1)} = & \left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k)}\right) \Big/ a_{ii} \\ i = & 1,2,\cdots n; k = 0,1,\cdots (iteration \quad index). \end{cases}$$

This is called the **Jacobi** iteration.

If, instead, only the lower triangular part $D+L$, is kept on the left, the pseudo-triangular form $(D+L)X = -UX + B$, leads to the iteration from which $(D+L)X^{(k+1)} = -UX^{(k)} + B$ from which $DX^{(k+1)} = -LX^{(k+1)} - UX^{(k)} + B$ and therefore $X^{(k+1)} = -D^{-1}\left(LX^{(k+1)} + UX^{(k)}\right) + D^{-1}B$. The corresponding component-wise formula is

$$\begin{cases} X^{(0)} = & \left(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}\right)^T \\ x_i^{(k+1)} = & \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right) \Big/ a_{ii} \\ i = & 1,2,\cdots n; k = 0,1,\cdots \left(iteration \quad index\right). \end{cases}$$

This is called the **Gauss–Seidel** iteration.

For convenience we recall several standard convergence criteria.

Lemma 1[9]

(1)The Jacobi iteration converges if and only if the spectral radius $\rho(J) < 1$, where $J = -D^{-1}(L+U)$.

(2) The Gauss–Seidel iteration converges if and only if the spectral radius $\rho(G) < 1$, where $G = -(D+L)^{-1}U$.

Lemma 2[9]

(1) If A is strictly diagonally dominant, then both the Jacobi and the Gauss–Seidel iterations converge

(2) If A is weakly diagonally dominant and irreducible, then both the Jacobi and the Gauss–Seidel iterations converge.

Lemma 3[9] Suppose A is symmetric and $a_{ii} > 0 (i = 1,2,\cdots,n)$. Then

(1) The Jacobi iteration converges if and only if A and $2D - A$ are both positive definite;

(2) The Gauss–Seidel iteration converges if A is positive definite.

2 MAIN RESULTS AND THEIR PROOFS

Multiplying both sides of (1) by A^T gives $A^T AX = A^T B$, Set $A_1 = A^T A, B_1 = A^T B$. The original system $AX = B$ is thus transformed into $A_1 X = B_1$. We now present a convergence theorem for the Jacobi and Gauss–Seidel iterations applied to this new system $A_1 X = B_1$.

Theorem: Let A be invertible (i.e. full rank). Then

(1) The Jacobi iteration for $A_1 X = B_1$ converges if and only if A_1 is invertible and $2D_1 - A_1$ is positive definite, where D_1 is the diagonal part of A_1.

(2) The Gauss–Seidel iteration for $A_1 X = B_1$ converges if A is invertible.

Proof :

(1)Because A is invertible, then A_1 is positive definite, hence all its diagonal entries are positive real numbers, and D_1 is positive definite and invertible. Consider the similarity transformation $D_1^{-1} A_1 = D_1^{-1/2}\left(D_1^{-1/2} A_1 D_1^{-1/2}\right) D_1^{1/2}$. Since A_1 is positive definite, $S = D_1^{-1/2} A_1 D_1^{-1/2}$ is real symmetric positive definite as well. Consequently, all eigenvalues μ of $D_1^{-1} A_1$ are positive real numbers. The hypothesis that $2D_1 - A_1$ is positive definite implies that $D_1^{-1/2}(2D_1 - A_1)D_1^{-1/2} = 2I - S$ is positive definite. Hence the eigenvalues of S satisfy $2 - \mu > 0$, i.e., $\mu < 2$, thus $\mu \in (0,2)$. The iteration matrix of the Jacobi method for $A_1 X = B_1$ is

$$J_1 = -D_1^{-1}(L_1 + U_1) = -D_1^{-1}(A_1 - D_1) = I - D_1^{-1} A_1,$$

Therefore the eigenvalues of J are $1 - \mu$, which satisfy $|1 - \mu| < 1$, so $\rho(J_1) < 1$. By Lemma 1 the Jacobi iteration converges.

(2)Since A_1 is real symmetric positive definite, the Gauss–Seidel iteration matrix for $A_1X = B_1$ is $G_1 = -(D_1 + L_1)^{-1}U_1^T = -(D_1 + L_1)^{-1}L_1^T$. For any nonzero vector x, let $y = G_1x$, then $(D_1 + L_1)y = -L_1^T x$, and consequently

$$(D_1 + L_1)(y - x) = -L^T x - (D_1 + L_1)x = -(D_1 + L_1 + L_1^T)x = -A_1x.$$

Thus $A_1x = (D_1 + L_1)(x - y)$. Set $z = x - y$, then $A_1x = (D_1 + L_1)z$. Now consider

$$x^T A_1x - y^T A_1y = x^T A_1x - (x - z)^T A_1(x - z) = 2z^T A_1x - z^T A_1z.$$

Substituting $A_1x = (D_1 + L_1)(x - y)$ yields

$$x^T A_1x - y^T A_1y = 2z^T(D_1 + L_1)z - z^T A_1z = z^T(2D_1 + 2L_1 - A_1)z.$$

Using $A_1 = D_1 + L_1 + L_1^T$, we obtain

$$2D_1 + 2L_1 - A_1 = 2D_1 + 2L_1 - (D_1 + L_1 + L_1^T) = D_1 + L_1 - L_1^T = D_1 + (L_1 - L_1^T).$$

Since $L_1 - L_1^T$ is skew- symmetric, $z^T(L_1 - L_1^T)z = 0$, hence $x^T A_1x - y^T A_1y = z^T D_1z$.
Because A_1 is positive definite, D_1 is also positive definite. If $z = 0$, then $A_1x = 0$ would force $x = 0$ contradicting the assumption that $x \neq 0$, Thus for every nonzero x, we have $z \neq 0$ and $z^T D_1z > 0$, which means $y^T A_1y < x^T A_1x$, This strict inequality implies that $|\lambda| < 1$ for every eigenvalue λ of G_1; i.e., $\rho(G_1) < 1$. By Lemma 1 the Gauss–Seidel iteration converges.

3 NUMERICAL EXAMPLES

Remark: In all the examples below, the initial guess is taken as $(x_1^0, x_2^0, x_3^0) = (0,0,0)$, the maximum number of iterations is 1000, and the stopping tolerance is 10^{-4}.
Example 1:

$$\begin{cases} 10x_1 - x_2 - 2x_3 = 7.2, \\ -x_1 + 10x_2 - 2x_3 = 8.3, \\ -x_1 - x_2 + 5x_3 = 4.2. \end{cases}$$

The coefficient matrix is strictly diagonally dominant. Hence the classical Jacobi, Gauss–Seidel, as well as the modified Jacobi and modified Gauss–Seidel all converge to $(1.1, \quad 1.2, \quad 1.3)$ taking 10, 6, 23 and 13 iterations, respectively.
Example 2:

$$\begin{cases} \sqrt{\dfrac{5}{2}}x_3 = 1 \\ \sqrt{3}x_1 + \dfrac{1}{\sqrt{3}}x_2 + \dfrac{1}{\sqrt{3}}x_3 = 2 \\ \sqrt{\dfrac{8}{3}}x_2 + \dfrac{1}{\sqrt{6}}x_3 = 3 \end{cases}$$

Here A is not strictly diagonally dominant, and $2D - A$ is not symmetric positive definite. However, $A_1 = A^T A$ is symmetric positive definite, and $2D_1 - A_1$ is also symmetric positive definite. Consequently, the plain Jacobi and Gauss–Seidel iterations do not converge for the original system, whereas the modified Jacobi and modified Gauss–Seidel iterations (applied to $A_1X = B_1$) both converge. The modified Jacobi iteration gives

$$(x_1^1, x_1^1, x_3^1) = (1.1547, \quad 2.0179, \quad 1.3202) \cdots\cdots (x_1^{25}, x_2^{25}, x_3^{25}) = (0.3842, \quad 1.6790, \quad 0.6325)$$

The modified Gauss–Seidel iteration gives

$$(x_1^1, x_2^1, x_3^1) = (1.1547, \quad 1.6330, \quad 0.3910) \cdots\cdots (x_1^8, x_2^8, x_3^8) = (0.3842, \quad 1.6790, \quad 0.6325)$$

4 CONCLUSION

For matrices that are not symmetric positive definite, this paper employs a matrix transformation $A \to A^T A$ to obtain a symmetric positive definite system. Under the symmetric positive definite condition a new convergence theorem for the Jacobi and Gauss–Seidel iterations is established and proved. Numerical experiments confirm the effectiveness of the theorem.

COMPETING INTERESTS

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